The Pentagon Project

Robert Geretschläger
Graz, Austria

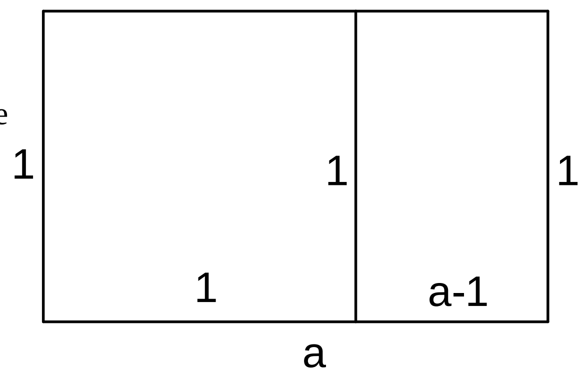
presented at MAKOS 2009, Janske Lazne, October 7th – 10th 2009

1. Introduction

The subject of the pentagon and the golden section is quite popular, and often dealt with in school at many age levels and at various levels of complexity. Applying methods of paper folding, some new and interesting aspects of this well known topic result, that can yield some very interesting and challenging classroom work.
Of course, this is only one of many topics in geometry and algebra that can be dealt with through the lens of paper folding. Many more such topics can be found in my book Geometric Origami (Arbelos Publishing, Shipley, UK), which is available at www.arbelos.co.uk.

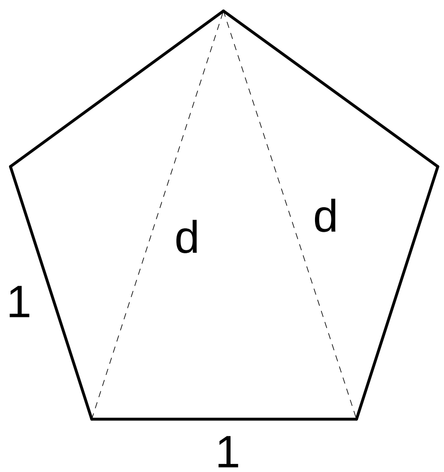
2. The Golden Section and the Regular Pentagon

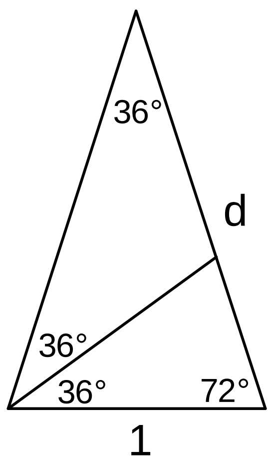
A standard method of defining the golden section is as the ratio ϕ of the lengths of the sides of rectangle with the property that a similar rectangle to the original results from cutting off a square. Such a rectangle is typically referred to as a *golden rectangle*. If we name the sides of the rectangle a and 1, this means a : 1 = 1 : (a-1), which is equivalent to $a^2 - a - 1 = 0$ or



$$\phi = a:1 = \frac{\sqrt{5}+1}{2}.$$

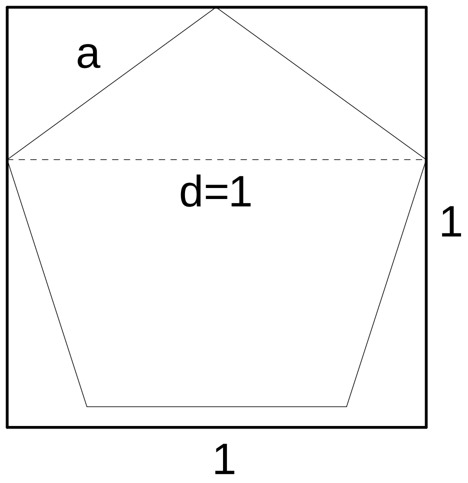
If we now take a close look at the regular pentagon, we see that a very similar situation results simply by drawing two of its diagonals. Simple calculation shows us that the interior angles of the pentagon are each equal to 108°. An isosceles triangle with the base of length 1 (the side of the regular pentagon, which we assume for the moment to have unit length) and sides of length d (the diagonals of the regular pentagon) must therefore have interior angles of 36° at the vertex and 72° at the base. Such a triangle is typically referred to as a *golden triangle*. Drawing the angle bisector in one of the 72° angles therefore splits the triangle into two smaller isosceles triangles, each with sides of length 1. Since the smaller of these is similar to the original triangle, and this triangle has sides of length 1 and a base of length d – 1, we once again obtain the relationship d : 1 = 1 : (d-1), and therefore $\phi = d:1$. The ratio of the length of the





diagonal of the regular pentagon to the length of its side is therefore also equal to the golden section ϕ.

3. Placing the Pentagon on the Paper



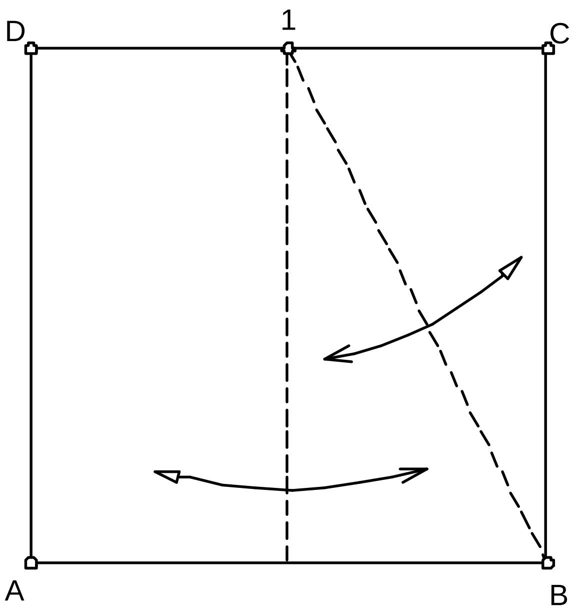
In origami, it is typical to fold squares of paper. If we wish to fold a pentagon from a square, we must therefore consider how the measurements of the square and the pentagon relate. In order to do this, we redefine our unit length as the length of the side of the folding paper, i.e. we consider the folding paper to be a unit square.
By placing the pentagon on the square as shown, we are assuming that the length of the diagonal of the pentagon is equal to 1. Setting a as the length of the side of the pentagon, we therefore get

$$a=\frac{a}{d}=\frac{1}{\phi}=\frac{2}{\sqrt{5}+1}=\frac{2\cdot\left(\sqrt{5}-1\right)}{\left(\sqrt{5}+1\right)\cdot\left(\sqrt{5}-1\right)}=\frac{2\cdot\left(\sqrt{5}-1\right)}{4}=\frac{\sqrt{5}-1}{2}.$$

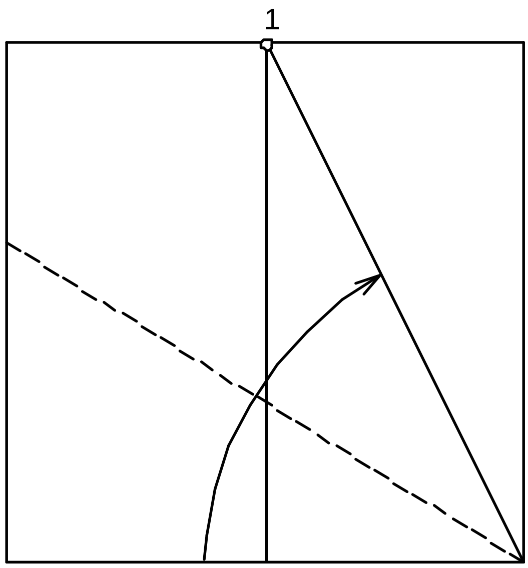
4. Folding the Pentagon

We are now ready to fold the pentagon. When this method is shown in a classroom situation, it is useful to have the students calculate the lengths of all line segments with respect to the unit length. It then becomes immediately obvious why this method indeed yields a theoretically precise regular pentagon, even if the students' folding may not be as precise as they might wish.

Step 1

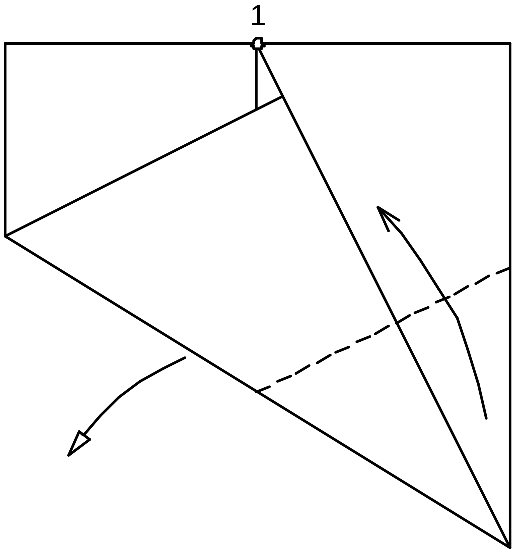


Step 2

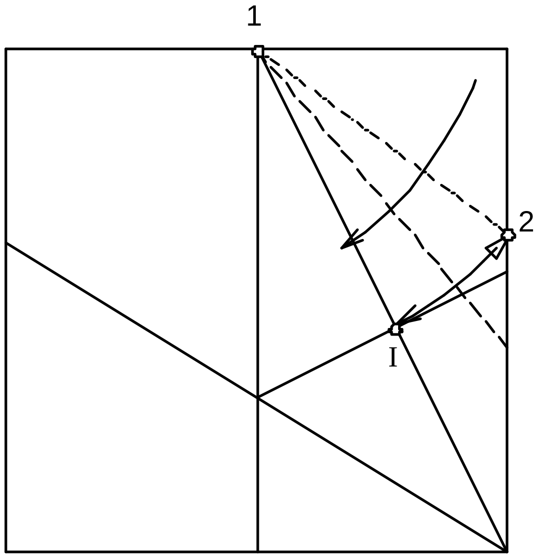


Step 3

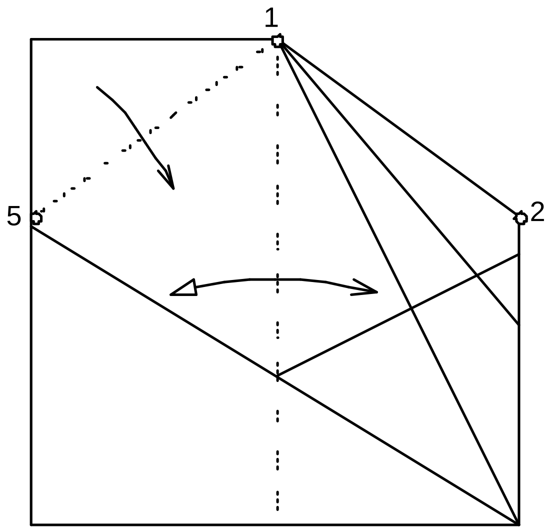
Step 4



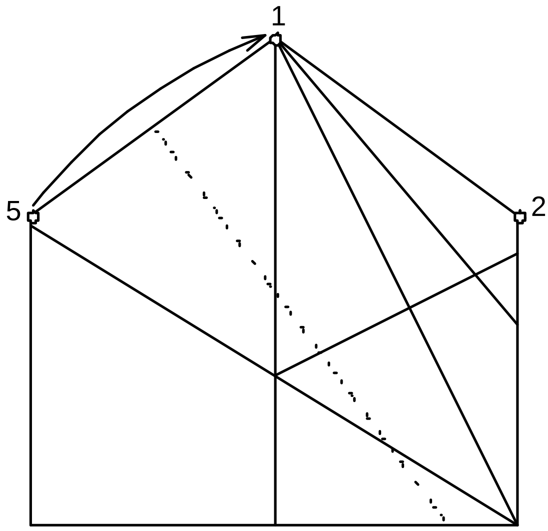
Step 5



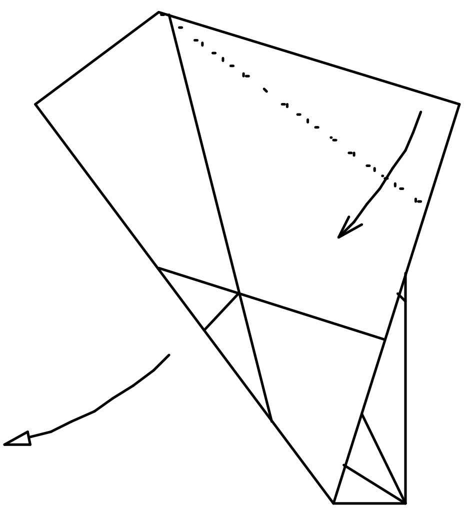
Step 6

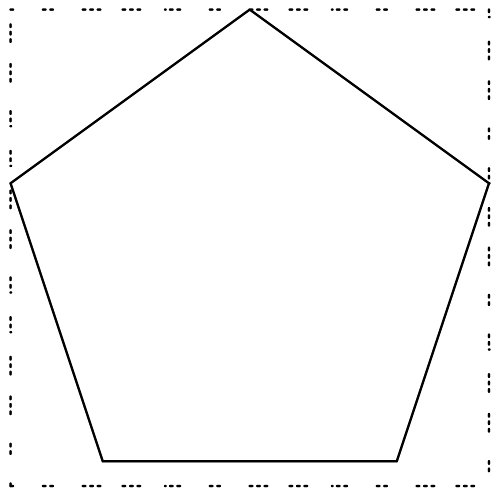


Step 7



Step 8





5. Additional Challenges for Advanced “Pentagonists”

If we are working with real students, some will inevitably be faster and more interested than others in the group. In order to keep these busy, there are numerous interesting questions available for further study. Some examples are the following:

- Can a regular pentagon with sides *a* longer than $1/\phi$ be placed in the interior of a unit square?

- Determine a folding sequence for a larger regular pentagon.

- Determine the largest possible value of *a*. Prove that your value is the largest possible.

Of course, the answer for the first question is yes, since the other two questions would otherwise be uninteresting. It is easy to draw such a pentagon. It is however not so easy to find a folding method for such a larger pentagon, and the proof of maximality is anything but easy. These are certainly problems for very advanced students, even if simply folding a regular pentagon from a square is accessible to all.

6. Conclusion

As mentioned at the beginning, this is only one example out of the many possibilities to demonstrate interesting facts in geometry and algebra to a school class using origami methods. A great deal of literature is available in book form and on the internet.

Happy folding!