Numerical Polyhedron Problems

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1 Introduction

Problems in solid geometry have become more and more rare in competitions over the years, reflecting developments in schools in most countries. Occasionally, we do still see problems in solid geometry, that tend to be either metric in nature (requiring the calculation of some volume, angle or distance) or combinatorial (counting vertices or faces, or involving coloring, for instance).

In this paper I would like to present some slightly different problems involving "numerical" properties of certain polyhedra, i.e. problems asking about relationships between the numbers of vertices, edges and faces of specific polyhedra. Some of these questions will ask for the smallest or largest such numbers possible under certain conditions, and many will ask about the existence of polyhedra with certain properties. For simplicity's sake, we will assume that all problems are restricted to convex polyhedra, although most can be stated just as easily with more general underlying conditions. The problems range in difficulty from very elementary (and accessible to very young students) to research level. While a few of these problems have actually been posed in competitions or journals, most are not too well known.

This paper was in fact inspired by the fact that a few interesting problems of this type had caught my eye in the last few years. Specifically, the following problems were either recent competition problems somewhere, or were posed in the problems section of an international mathematical journal.

Problem 1 *One face of a polyhedron is a pentagon. What is the smallest number of faces the polyhedron can have?*

A) 5 B) 6 C) 7 D) 8 E) 10

(Kangaroo competition 2002, Junior and Étudiant)

Problem 2 *A prism has 2002 vertices. How many edges does the prism have?*

A) 3003 B) 1001 C) 2002 D) 4002 E) 2001

(Kangaroo competition 2002, Étudiant)

Problem 3 *Suppose we want to construct a solid polyhedron using just n pentagons and some unknown number of hexagons (none of which need be regular), so that exactly three faces meet at every vertex on the polyhedron. For what values of n is this feasible?*

(Crux Mathematicorum with Mathematical Mayhem, April 2001, Problem H287, [1])

Problem 4 *Find all bounded convex polyhedra such that no three faces have the same number of edges.*

(The American Mathematical Monthly, Feb. 2001, Problem 10856, proposed by Andrei Jorza, [4])

These four problems illustrate the levels of difficulty quite well, since the first two are quite elementary, the third easy enough, but requiring some previous knowledge, and the fourth quite difficult. Their solutions will be given in the appropriate sections to come.

2 Elementary Problems

Let us consider the first two of these problems. They represent what is probably the easiest type of polyhedra problem, since they only require knowledge of those types of polyhedra that tend to be best known to students, namely pyramids and prisms.

An n-sided pyramid is, of course, a polyhedron with an n-gon as one face (the base) and n triangles as faces that all have a common vertex (the apex, Figure 1a). An n-sided pyramid has $n+1$ faces, $n+1$ vertices and $2n$ edges.

Similarly, an n-sided prism is a polyhedron with two congruent n-gons as faces (the lower and upper bases) and n parallelograms as faces joining these two (Figure 1b). An n-sided prism has $n+2$ faces, $2n$ vertices and $3n$ edges.

A further useful type of polyhedron in this context is the slightly less well known anti-prism. An n-sided anti-prism is a polyhedron with two congruent n-gons as faces (the lower and upper bases) and $2n$ triangles as faces, each of which has two corners in common with one of the bases and the third in common with the other. (Figure 1c). An n-sided anti-prism has $2n+2$ faces, $2n$ vertices and $4n$ edges.

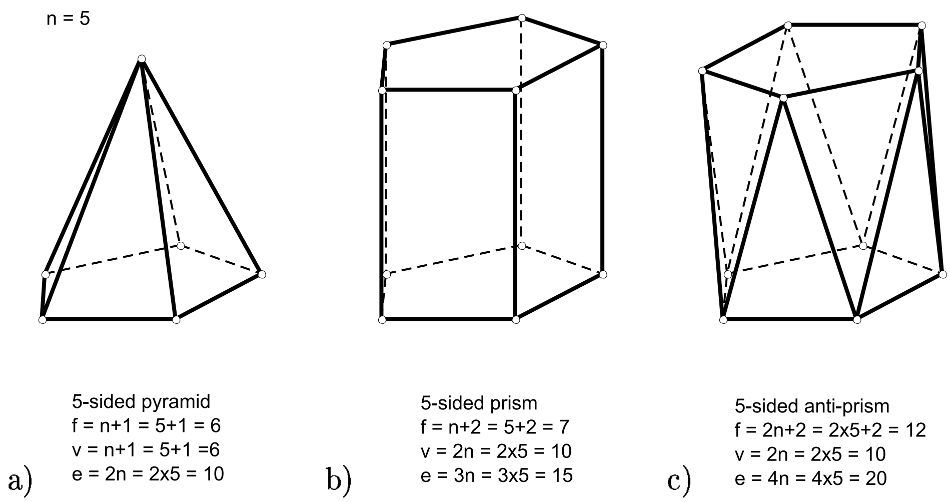


Figure 1

We also have some additional information at our disposal concerning the faces of these special types of polyhedra. An n-sided pyramid ($n > 3$) has one n-sided face and n triangular faces. An n-sided prism ($n \geq 3$, $n \neq 4$) has two n-sided faces and n 4-sided faces. Finally, an n-sided anti-prism ($n > 3$) has two n-sided faces and $2n$ triangular faces. In the special case $n = 3$, the pyramid is a tetrahedron (4 triangular faces) and the anti-prism is an octahedron (8 triangular faces). For $n = 4$ the prism is a parallelepiped (6 four-sided faces).

Armed with this elementary information, we can now take a closer look at the solutions of the first two problems.

Solution to Problem 1: An edge of a polyhedron is always a common side of two of its faces. If one face of a polyhedron is a pentagon, it must have at least one more face sharing each of the five sides of the pentagon. No two of these can be the same, since that would mean that the plane of such a face would pass through at least three of the corners of the pentagon, and must therefore be identical to the plane of the pentagon, contradicting the fact that two faces of a convex polyhedron cannot lie in the same plane. The polyhedron in question must therefore have at least 6 faces. Since we know that a 5-sided pyramid is indeed a polyhedron with 6 faces and a pentagonal face, the answer to the problem is B). qed

We note at this point, that it is not sufficient to know that a polyhedron with the required property must have at least 6 faces. We must demonstrate the existence of a specific polyhedron with exactly 6 faces in order to complete the proof. As we shall see, this aspect of problems of this sort tends to be the more difficult. In this case, our knowledge of pyramids helped us with the existence aspect of the proof.

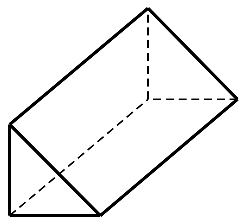
Solution to Problem 2: The solution to this problem merely requires the information about prisms we recalled earlier. We know that an n-sided prism has $2n$ vertices and $3n$ edges. If a prism has 2002 vertices, we have $n = 1001$, and the prism therefore has $3 \cdot 1001 = 3003$ edges. The answer to the problem is therefore A). qed

Many similar problems can easily be stated simply by changing the numbers in these problems. (In fact the same will hold for many of the following problems.) Another way to find similar, or at least analogous, problems is to exchange "faces" for "edges" or "vertices". In this paper, I will present a number of alternative problems, not always with solutions, but an implicit challenge to the reader is always present to find more similar problems to those stated. An example of such an analogous problem to Problem 1 is the following.

Problem 5 *A polyhedron has a 6-sided face. What is the smallest number of edges the polyhedron can have?* *A) 6* *B) 7* *C) 9* *D) 12* *E) 18*

(The answer is D), of course.)

An interesting, if easy, problem of a similar type was posed as problem 14 of the UK Senior Mathematical Challenge 2002:



Problem 6 *Which shape cannot be obtained as the cross-section (in any direction) of this solid, which is a triangular prism with three rectangular faces?*
A) triangle *B) rectangle* *C) trapezium* *D) pentagon* *E) hexagon*

Solution: The sides of the polygon resulting as a cross-section of the polyhedron must be the lines that the intersecting plane has in common with the planes of the faces of the polyhedron. Since the prism only has five faces, the cross-section cannot have more than five sides. The answer is therefore E). qed

Note that actually finding planes that yield the other four shapes as cross-sections is not necessarily easy. Trying to do so is quite a valuable exercise is spatial reasoning.

A number of problems can be derived by combining the elementary numerical properties of pyramids, prisms and anti-prisms. For instance, we can obtain new polyhedra by "gluing" together two of these elementary building blocks if we choose them such that they have a common k-sided face (Figure 2). The result is a polyhedron, the number of whose vertices, edges and sides can easily be stated. If the parameters for the original two polyhedra are v_1, e_1, f_1 and v_2, e_2, f_2 respectively, the resulting "glued" polyhedron has

$$v = v_1 + v_2 - k, \quad e = e_1 + e_2 - k \quad \text{and} \quad f = f_1 + f_2 - 2.$$

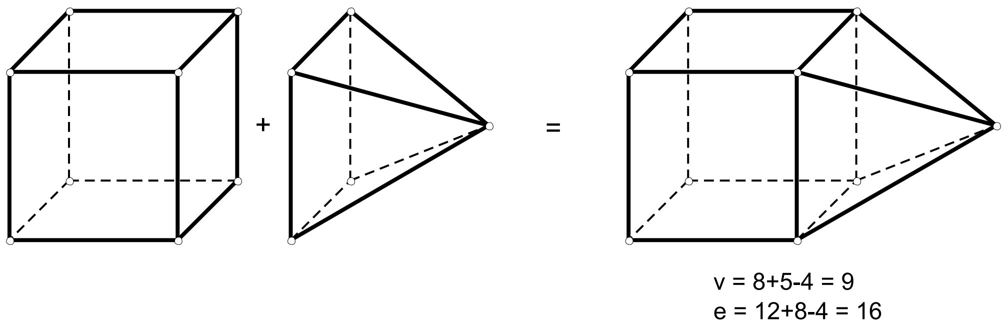


Figure 2

One problem using this idea in the proof is the following.

Problem 7 *A polyhedron has two n-sided faces* ($n > 3$) *and t triangular faces. How many of the following values of t are possible?*

$$1, 2, 3, 4, 5, 6, 7, 8$$

A) 2 *B) 3* *C) 4* *D) 5* *E) 6*

Solution: The answer is B), since t can be equal to 4, 6 or 8, bur not 1, 2, 3, 5 or 7. The proof of this is included in the following more general (and more difficult) version of the problem.

Problem 8 *A polyhedron has two n-sided faces* ($n > 3$) *and t triangular faces. How many of the positive integers i with* $1 \leq i \leq k$ *are possible values for t for any given positive integer k?*

Solution: If a polyhedron has a face with at least four sides, it must have at least five faces altogether by the reasoning used in Problem 1. We therefore have $f \geq 5$, and therefore $t = f - 2 \geq 3$.

t can never be an odd number. If we assume that a polyhedron has 2 n-sided faces and t triangular faces, the fact that each edge of the polyhedron is common to two faces means

$$e = \frac{1}{2} \cdot (2n + 3t) = n + \frac{3}{2} \cdot t,$$

but this is not an integer if t is odd.

t can however assume any even integer value greater than three. A polyhedron with two 4-sided faces and four triangular faces can be obtained as shown in Figure 3.

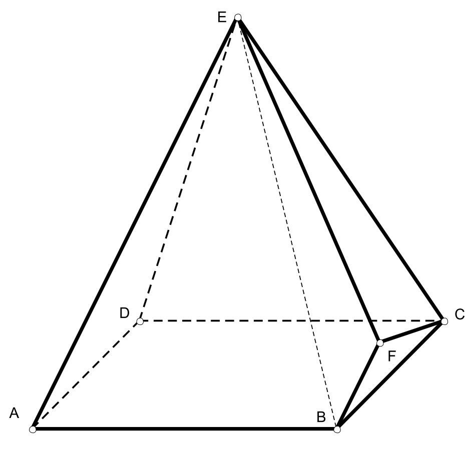


Figure 3

Starting with a regular quadratic pyramid $ABCDE$, a point F is chosen as shown, such that $BF \| AE$. The point F is then certainly in the plane determined by A, B and E, and the polyhedron $ABCDEF$ has two 4-sided faces and four triangular faces as required.

If we are given a convex polyhedron P with t triangular faces ($t > 1$), we can construct a convex polyhedron $\overline{P}$ with $t+2$ triangular faces in the following manner.

Let A, B and C be the corners of a triangular face of P, and D be the centroid of $\triangle ABC$. Further, let X, Y and Z be the points in which the line perpendicular to the plane of $\triangle ABC$ intersects the planes of the faces of P having edges AB, BC and CA respectively in common with $\triangle ABC$. (We assume that X, Y and Z are all finite points and all are on the opposite side of P with respect to the plane of $\triangle ABC$. If this is not the case for one or more of the points, we can replace the point or points in question by any random point on the appropriate side of the plane of $\triangle ABC$ without it affecting the validity of the construction.) If we chose E as the mid-point of the line segment joining D to the point among X, Y and Z closest to D, E is certainly on the opposite side of P with respect to $\triangle ABC$. Furthermore, P must lie completely on one side of the plane joining A, B and E, since this is true for the planes of both faces of P through AB and E was chosen between D and X, Y and Z. The same holds for the planes joining B, C and E and C, A and E respectively.

We define a polyhedron $\overline{P}$, whose faces are all identical to those of P with the exception of $\triangle ABC$, which is replaced by the new faces $\triangle ABE$, $\triangle BCE$ and $\triangle CAE$. $\overline{P}$ is then certainly convex, and has $t+2$ triangular faces as required.

We see that all even positive integer values for i greater than 3 are possible values for t, and the answer to the question is therefore

$$0 \quad \text{if} \quad 1 \leq k \leq 3 \quad \text{and} \quad \left[\frac{k}{2}\right] - 1 \quad \text{if} \quad k > 3.$$

qed

A somewhat surprising result related to this idea is the following. (Thanks go to Ingmar Lehmann for communicating this problem to me.)

Problem 9 *We are given a quadratic pyramid and a triangular pyramid. All edges of both pyramids are the same length a. We glue one of the faces of the triangular pyramid completely onto one of the triangular faces of the quadratic pyramid. How many vertices, edges and faces does the resulting "glued" polyhedron have?*

Solution: For the triangular pyramid (tetrahedron), we have $v = 4$, $e = 6$ and $f = 4$, and for the quadratic pyramid we have $v = 5$, $e = 8$ and $f = 5$. By the ideas explained above, we would expect the solution to this problem to be

$$v = 4 + 5 - 3 = 6, \quad e = 6 + 8 - 3 = 11 \quad \text{and} \quad f = 4 + 5 - 2.$$

The result for the number of vertices $v = 6$ is indeed correct by the reasoning stated above, but the results for e and f are not.

The somewhat surprising reason for this is the fact that two pairs of triangular faces of the two pyramids end up in the same plane after "gluing", resulting in two rhombic faces of the "glued" polyhedron (see Figure 4).

We can see this by adding a line segment EF to the quadratic pyramid $ABCDE$ (with square face $ABCD$), such that $EF \| AB \| CD$ and $|EF| = |AB| = |CD|$. Since $EF \| AB$, all points A, B, E and F lie in a common plane, and we have

$$\angle EBA = \angle BEF = 60^o.$$

Since $|EF| = |AB| = |EB|$, we see that $\triangle EFB$ is equilateral. The same holds for $\triangle EFC$, and since $\triangle EBC$ must therefore also be equilateral, $EFBC$ is a regular tetrahedron.

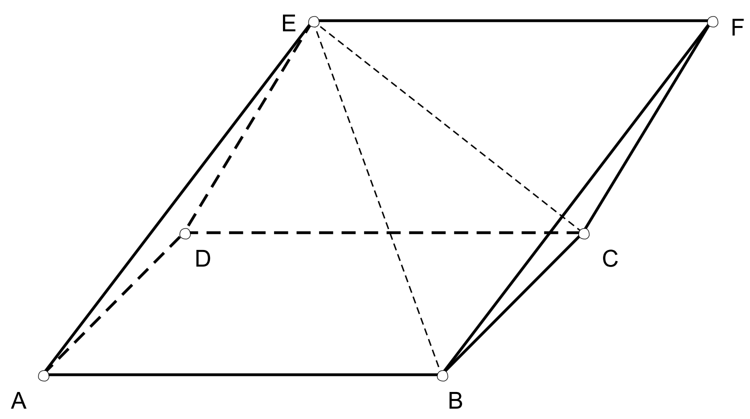


Figure 4

Gluing the tetrahedron $EFBC$ onto the quadratic pyramid $ABCDE$ therefore yields a polyhedron with triangular faces ADE and BCF and 4-sided faces $ABCD$ (square) and $ABFE$ and $DCFE$ (rhombic), The polyhedron therefore, somewhat surprisingly, has 5 faces and 9 edges ($f + v = e + 2 \;\rightarrow\; 5 + 6 = 9 + 2$). qed

We can note that this proof also includes the following result.

Problem 10 *We are given a quadratic pyramid with all eight edges of equal length and a regular tetrahedron. Prove that the angle between an edge and a face of the tetrahedron is equal to the dihedral angle of the square pyramid at any of the edges of the square face.*

We now come to a slightly different type of problem, the first of which is the following.

Problem 11 *A polyhedron P has a 5-sided face and a 4-sided face. These two faces do not have a common edge. What is the smallest number of edges P can have?*

Solution: The smallest number of edges is 14. Since the 4- and 5- sided faces do not have a common edge, there must be at least one additional edge through each of the 5 vertices of the pentagonal side. The polyhedron therefore cannot have less than $5 + 4 + 5 = 14$ edges.

That such a polyhedron is possible can be shown in many ways, but two such polyhedra can be derived from a cube or a regular 5-sided prism as shown in Figure 5.

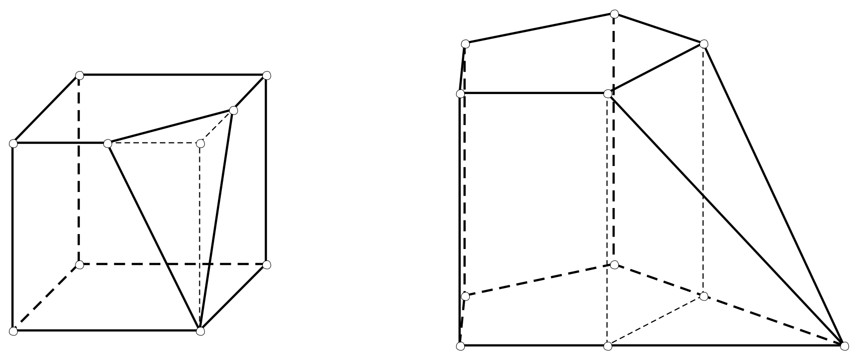


Figure 5

As already mentioned, further problems can be derived from problems of this type by exchanging the variables we ask to minimize. An example is the following.

Problem 12 *A polyhedron P has a 5-sided face and a 4-sided face. These two faces do not have a common edge. What is the smallest number of faces P can have?*

The answer here is 7. The solution is essentially the same as that for the preceding problem.

A very similar problem is also the following.

Problem 13 *A polyhedron P has a 5-sided face and a 6-sided face. What is the smallest number of faces P can have?*

Solution: Since one of the faces of P is 6-sided, P must have at least 7 sides. One 7-sided polyhedron with the reuired property is pictured in Figure 6.

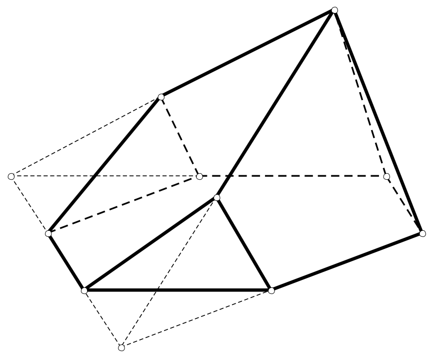


Figure 6

The answer is therefore 7. qed

A useful concept for this type of problem is that of a "k-hedral vertex". Still restricting ourselves to convex polyhedra for simplicity's sake, we say that a vertex V of a polyhedron P is "k-hedral" if exactly k faces of P share a common corner in V. Since any two neighboring faces among these determine one edge of P with an end-point in V, k is also the number of edges of P with V as an end-point.

We can immediately use this concept to formulate "dual" problems to problems involving k-sided faces. The "dual" of any problem results from the original by exchanging the concepts of "face" and "vertex" (and simultaneously "common edge of two faces" and "edge joining two vertices"). The dual problem to Problem 1, for instance, is the following.

Problem 14 *One vertex of a polyhedron is pentahedral. What is the smallest number of vertices the polyhedron can have?*

The answer to this problem, as to its dual, is again 6. (Note that alternative expressions for "k-hedral" are used for small values of k. We generally use "trihedral" for "3-hedral", "tetrahedral" for "4-hedral", and similarly "pentahedral" and "hexahedral". Note also that the concept of "duality" as expressed here is neither very general nor very precise. It is however sufficient for the purpose of developing problems under the limited conditions we have imposed here.)

Not only does the concept of the k-hedral vertex help us formulate dual problems, we can also combine conditions on k-hedral vertices and n-sided faces to produce new problems. A few examples of this kind are as follows.

Problem 15 *Determine the smallest number of edges a polyhedron P can have if it is known to have a 5-sided face and a pentaherdral vertex.*

Solution: If P has a 5-sided face, there must be at least three edges with end-points in each of the corners of the 5-sided face, two of which can be sides of the pentagonal face. There must therefore be at least one edge with an end-point in each corner of the pentagonal face beside the sides of the face, and P must therefore have at least $5+5=10$ edges. A 5-sided pyramid does indeed have a 5-sided face and 10 edges, and since its apex is a pentahedral vertex, such a pyramid is indeed a polyhedron with the required properties. 10 is therefore the smallest number of edges of such a polyhedron. qed

Problem 16 *Determine the smallest number of edges a polyhedron P can have, if it is known to have a 5-sided face and a tetrahedral vertex.*

Solution: Since the tetrahedral vertex can either be one of the corners of the 5-sided face or not, we must consider both of these cases.

If the tetrahedral vertex V_4 is a corner of the 5-sided face, P must have at least 7 vertices, since the 5-sided face has 5 corners and two of the edges with end-points in V_4 must also have end-points which are not corners of the 5-sided face. If V_4 is not a corner of the 5-sided face, P must also have a seventh vertex, since there must be an edge through each of the 5 corners of the 5-sided face which is not a side of that face, and not all 5 of these can have end-points in V_4, since V_4 is tetrahedral. In either case, we see that P must have at least 7 vertices.

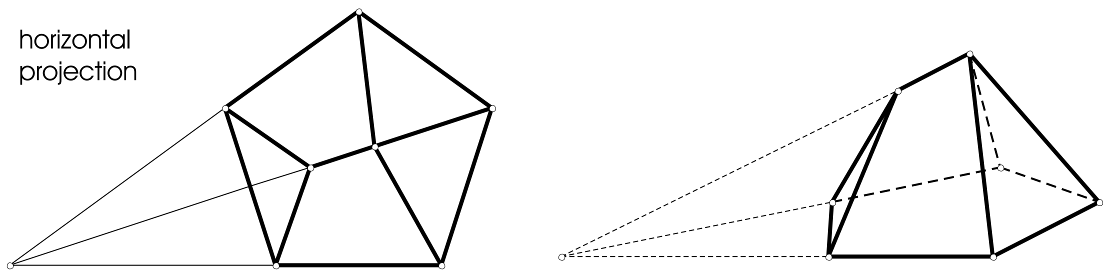


Figure 7

A well-known inequality of convex polyhedra (which I will discuss in more detail in section 3) states that

$$e \geq \frac{3}{2}v$$

must hold. Since $v \geq 7$, we have $e \geq \frac{21}{2}$, and therefore $e \geq 11$. There is indeed a polyhedron with the required properties with $e = 11$ as we see in Figure 7, and it follows that the required number is 11. qed

In this problem, the existence of a polyhedron with the required properties was shown by constructing one in various views. As was stated before, the explicit construction of a polyhedron with required properties is often the most difficult part of solving this type of problem. Perhaps this is a reason that problems like this are not more popular, since constructions of solids are not commonly studied in any depth in most countries.

One way to avoid this, which is however much more advanced from a theoretical standpoint, is to allow Schlegel diagrams in such proofs. Schlegel diagrams are graphs associated with polyhedra. The vertices and edges of the graphs correspond to the vertices and edges of the polyhedra. There is a fairly deep theorem in polyhedron theory stating that any 3-connected graph can be realized as a convex polyhedron and that the graph associated with a polyhedron is always 3-connected. 3-connectedness can be naively described as the property of any two vertices being connected by three distinct paths. It can be shown that this is equivalent to the property that any two vertices of the graph can be removed, along with all edges having either as an end-point, without disturbing the connectedness of the graph. (Figure 8)

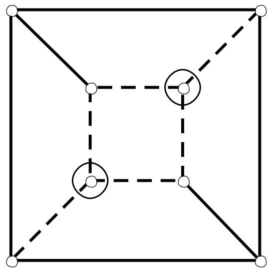


Figure 8

It follows that all Schlegel diagrams are 3-connected graphs and vice versa. (For more information, see [3].)

In Figure 7, we can interpret the horizontal projection of P as its Schlegel diagram, since the projections of the two vertices that are not corners of the 5-sided face lie inside the projection of this face.

In this paper, I will mostly show projections of polyhedra to illustrate their existence, since I believe this to be more elementary, but also more fun. While there are many complex results in graph theory relevant to the study of polyhedra, I am attempting to keep them out of this discussion as far as possible. In the next problem, I would however like to give an example of how a graph-theoretical approach can lead to a solution of a problem of this type.

Problem 17 *Determine the smallest number of edges a polyhedron P can have if it is known to have a 5-sided face and a pentahedral vertex in one of the corners of the 5-sided face.*

Solution: As stated, we shall consider the Schlegel diagram for this solution. In Figure 9a we see the 5-sided face as the outer polygon of the diagram. One of the corners A is pentahedral, so there are another 3 edges passing through this corner, apart from the sides of the pentagon. We name the other end-points of these edges F, G and H as shown. P must have at least 8 vertices, and due to the inequality

$$e \geq \frac{3}{2} \cdot v \geq 12,$$

we see that P must have at least 12 edges. In fact, P must have more than 12 edges, however. If P had exactly 12 edges, it could not have more than 8 vertices, since this would mean $e \geq \frac{3}{2} \cdot 9 = \frac{27}{2}$. In this case, P would have at least 14 edges. If we assume that P has the 8 vertices shown, each of the vertices (other than A) must be at least trihedral, and since each edge has two end-points, this means $e \geq \frac{1}{2} \cdot (5 + 3 \cdot 7) = 13$. P must therefore have at least 13 edges.

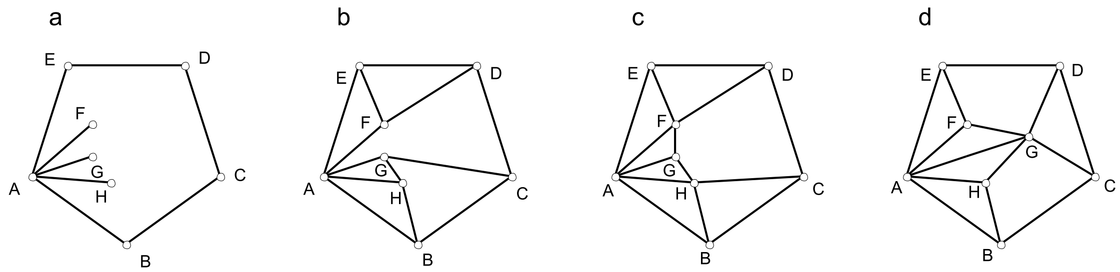


Figure 9

This is not possible either, however. Let us assume that P has 13 edges. We shall try to add the missing 5 edges to the 8 already in Figure 9a. One each must pass through each of the trihedral vertices B, C, D and E. E must be joined with F, since joining E with G or H would leave nothing to join F with. (The edges may not cross each other if they do not have a common vertex in their common point.) Similarly, B must be joined with H. This would leave us with 3 more edges, two of which must originate in C and D, and two of which must pass through G. One of C and D must be joined to G, and one to one of the other two inner points, say F.

The final edge must then join G and H in order for both to be trihedral, but that means that A, F, D, C and G must lie in a common plane, which is not possible if P is to be a polyhedron (Figure 9b). The only way to solve this problem is by adding another edge, which can be done in a number of ways (Figure 9c, 9d), and we see that the smallest number of edges fulfilling all stated requirements is 14. qed

3 Some Properties of Polyhedra

In order to deal with the solutions to more complex problems, it will help us to remind ourselves of some important relationships pertaining to numerical properties of polyhedra. In order to keep things simple, we will continue to restrict ourselves to convex polyhedra, although it would be enough here to restrict ourselves to polyhedra with genus 0 (i.e. no "holes").

For such polyhedra, perhaps the most important numerical relationship is

$$\textbf{EULER's formula}: \ v + f = e + 2,$$

wherby v denotes the number of vertices of the polyhedron, f the number of faces and e the number of edges.

There are many proofs of this result, for instance in [2] or [5].

There are also a number of slightly less well known relationships between v, f and e that can make interesting problems in their own right, assuming that they are not already well known to students. (They can be found in the literature, for instance in [5]).

Problem 18 *Prove that*

$$3f \leq 2e \quad \text{and} \quad 3v \leq 2e$$

hold for all convex polyhedra.

Solution: Every face of a polyhedron P has at least 3 sides. Let f_k be the number of faces of P with k sides. Since each edge of P is a common side of two of the faces of P, we have

$$e = \frac{1}{2} \cdot (3f_3 + 4f_4 + 5f_5 + \ldots),$$

and therefore

$$\begin{aligned} 2e &= 3f_3 + 4f_4 + 5f_5 + \ldots \\ &\geq 3f_3 + 3f_4 + 3f_5 + \ldots \\ &= 3f, \end{aligned}$$

proving the first inequality. The other can be proven analogously, substituting the number of edges through each vertex for the number of sides of each face. qed

Problem 19 *Prove that*

$$e + 6 \leq 3f \quad \text{and} \quad e + 6 \leq 3v$$

hold for all convex polyhedra.

Solution: In the preceding problem, we saw that

$$3v \leq 2e$$

holds. Euler's formula states

$$v = e - f + 2,$$

and substituting for v yields

$$3 \cdot (e - f + 2) \leq 2e,$$

or

$$3e - 3f + 6 \leq 2e,$$

which is equivalent to

$$e + 6 \leq 3f,$$

proving the first inequality. The second is obtained by analogously substituting

$$f = e - v + 2$$

in the inequality

$$3f \leq 2e.$$

qed

Problem 20 *Prove*

$$v + 4 \leq 2f \leq 4v - 8 \quad \text{and} \quad f + 4 \leq 2v \leq 4f - 8.$$

Solution: By Euler's formula, we have

$$2v + 2f = 2e + 4.$$

Since $3f \leq 2e$ and $3v \leq 2e$ hold, we have

$$2v + 2f \geq 3f + 4 \quad \Rightarrow \quad 2v \geq f + 4$$

and

$$2v + 2f \geq 3v + 4 \quad \Rightarrow \quad 2f \geq v + 4.$$

Multiplying by 2 yields

$$2f \leq 4v - 8 \quad \text{and} \quad 2v \leq 4f - 8.$$

qed

Armed with this knowledge, we can now readily solve Problem 3.

Solution to Problem 3: Let m be the number of hexagonal faces of polyhedron P (recalling that n denotes the number of pentagonal faces). P then has

$$f = m + n$$

faces. Since each edge is shared by two faces, the number of edges of P is

$$e = \frac{6m + 5n}{2},$$

and since each edge joins two vertices and three edges pass through each vertex, the number of vertices of P is

$$v = \frac{2}{3} \cdot e = \frac{6m + 5n}{3}.$$

By Euler's formula, we have

$$v + f = e + 2,$$

and substituting yields

$$\frac{6m+5n}{3} + (m+n) = \frac{6m+5n}{2} + 2,$$

which simplifies to

$$n = 12.$$

qed

We can note that the regular dodecahedron and the "soccer ball" (truncated icosahedron) are polyhedra of this type, with $m = 0$ and $m = 20$ respectively.

4 Polyhedra with "Very Different" Faces

Let us recall that problem 4 in Section 1 asks us to "find all bounded convex polyhedra such that no three faces have the same number of edges". This is, of course, a very interesting problem in itself, and we shall consider its solution in a moment. First, however, we note that this problem suggests a whole category of numerical polyhedra problems, namely problems concerning polyhedra with "very different" faces. We can take this to (loosely) mean polyhedra with as many faces with different numbers of sides as possible. (To clarify the terminology, we shall use the term "edge" when it applies to a bounding line segment of a polyhedron, but "side" when it applies to a bounding line segment of a face of the polyhedron.)

A few such problems are as follows.

Problem 21 *Does a polyhedron exist, no two of whose faces have the same number of sides?*

Solution: No such polyhedron can exist. In order to see this, we assume that one does exist, and then note that one specific face of the polyhedron must have more sides than any other, since each face has a different number of sides. Let n_{max} be this maximum number. Since some other face of the polyhedron must an edge in common with the n_{max}-sided face in each of its sides, and no two faces of the polyhedron can have more than one side in common, we see that $f \geq n_{max} + 1$ must hold. On the other hand, the number of sides of each of the faces of the polyhedron must be no less than 3 an no more than n_{max}. Since no two faces have the same number of sides, it follows that $f \leq n_{max} - 2$ holds, and we have a contradiction. qed

This result immediately implies its dual:

Problem 22 *Does a polyhedron exist, no two of whose vertices are the end-points of the same number of edges?*

(This was problem number 4 in round 38 of the International Mathematical Talent Search. There, it was stated as follows: Prove that every polyhedron has two vertices at which the same number of edges meet.)

Solution: Since the existence of such a polyhedron would imply the existence of its dual, which was proven not to exist in the previous problem, there can be no such polyhedron. Another way to see this is to retrace the steps of the previous solution, replacing the term "face" by "vertex" and f by v. qed

We can note that these proofs also show us that there can be no polyhedra with exactly one pair (or triple) of faces with the same number of sides, and otherwise no such pair of sides. Nor can there exist a polyhedron with exactly two such pairs of faces. (The analogous claims hold for the vertices.) Some further problems suggested by these results are therefore the following:

Problem 23 *Does a polyhedron exist with exactly one pair of faces with an equal number of sides?*

Problem 24 *What is the smallest number m such that there exists a polyhedron with $f = k+m$ faces, k of which can be chosen such that no two of these k have an equal number of sides?*

Solution: The contradiction used in Problem 21 will always hold for $m < 3$. We see that $m \geq 3$ must hold, and there are many examples for polyhedra with $m = 3$, such as those shown in Figure 10. qed

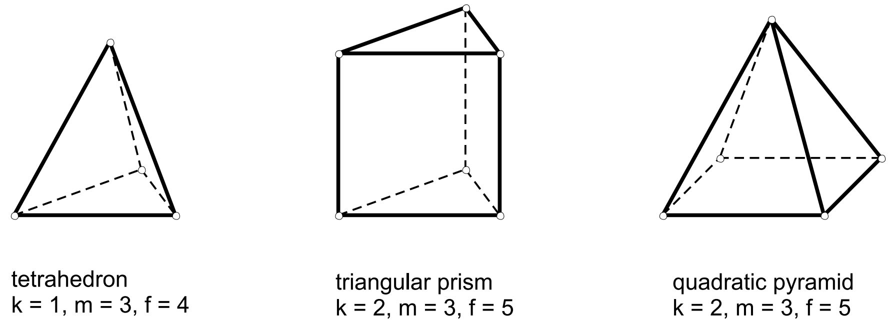


Figure 10

Note that $f \geq 2k$ holds for all these polyhedra. The following question arises:

Problem 25 *Let f be the number of faces of a polyhedron and k be the largest number of faces that can be chosen such that no two of the chosen faces have an equal number of sides. Is it always true that $f \geq 2k$ must hold?*

The somewhat surprising (for me) answer in no, as we can see by taking a look at the following example (due to Gottfried Perz).

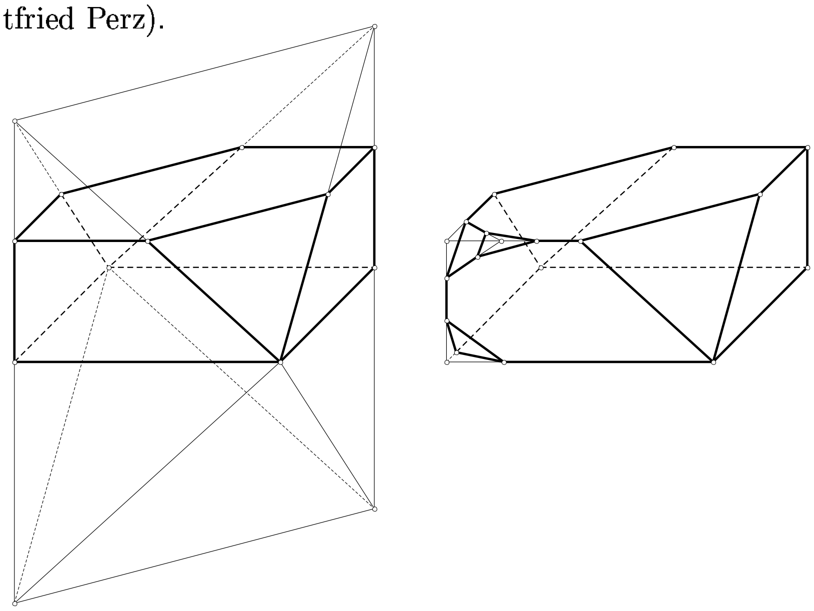


Figure 11

Starting with half of an octahedron, we cut off the top, parallel to the 4-sided face as shown in the left part of Figure 11. The result is a polyhedron with 5 four-sided faces, 2 triangular faces and a hexagonal face. Successively cutting off three vertices as shown in the right part of Figure 11 yields a polyhedron with 11 faces, one each of which has $5, 6, 7$ and 8 sides, 3 of which have 4 sides, and 4 of which have 3 sides. For this polyhedron, we have $f = 11$ and $k = 6$, and since $11 < 2 \cdot 6$, we see that this example contradicts $f \geq 2k$. qed

The next question that naturally arises from this is the following:

Problem 26 *Let f be the number of faces of a polyhedron and k be the largest number of faces that can be chosen such that no two of the chosen faces have an equal number of sides. Determine the largest possible lower bound for the parameter $a = \frac{f}{k}$.*

Unfortunately, I am not yet aware of a full solution to this problem. An obvious lower bound for a is 1, but as we know from Problem 21, this value is not attainable, since all sides of a polyhedron cannot have different number of sides. The value of a for the example shown in Figure 11 is $\frac{11}{6}$, and the value of a for the following object (also due to Gottfried Perz) is $\frac{8}{5}$.

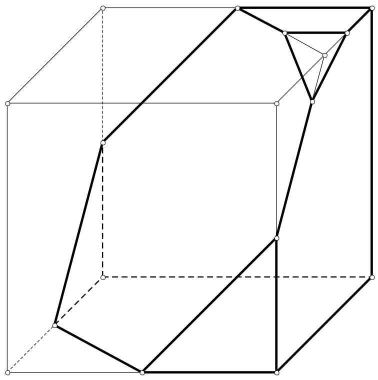


Figure 12

The object is half of a cube (the cube has been cut through its mid-point perpendicular to its diagonal), with a vertex cut off as shown. Such an object has 8 faces, of which 3 are 3-sided, 2 5-sided, and one each 4-,6- and 7-sided. Perhaps readers of this paper will be inspired to further improve on this lowest value for a, or even to find the largest lower bound for all possible values of a. (Experimenting with such polyhedra seems to indicate that $f \geq 2k - 2 \Leftrightarrow \frac{f}{k} \geq 2 - \frac{2}{k}$ must hold. If this is true, $\frac{8}{5}$ is indeed the smallest attainable value of a, since we can show that $k = 4$ implies $f \geq 7$ and $k = 3$ implies $f \geq 6$.)

We are now ready to turn our attention to Problem 4. A simpler version of the problem is the following:

Problem 27 *Find all bounded convex polyhedra with the following property: If n_{max} is the maximum number of sides of any of the faces of the polyhedron, the polyhedron has exactly two faces with $3, 4, \ldots, n_{max}$ sides each.*

Solution: To simplify notation, let $x = n_{max} - 2$. Since any polyhedron with the required property has two faces with $3, 4, \ldots, (x + 2)$ sides, the total number of faces f must be equal to $2x$. The number e of edges of such a polyhedron must be equal to

$$\begin{aligned} e &= \frac{1}{2} \cdot 2 \cdot (3 + 4 + \ldots + (x + 2)) \\ &= \frac{x(x + 5)}{2}, \end{aligned}$$

and since each edge is bounded by two vertices and each vertex is end-point of at least three edges, the number of vertices of the polyhedron must satisfy the condition

$$v \leq \frac{2}{3} \cdot e = \frac{x(x + 5)}{3}.$$

(Note that the inequality $3v \leq 2e$ was part of Problem 18.)
Since all polyhedra with the required property certainly satisfy Euler's formula, we have

$$v = e + 2 - f.$$

This means that

$$v = \frac{x(x+5)}{2} + 2 - 2x \leq \frac{x(x+5)}{3}$$

or

$$\begin{aligned} 3x^2 + 15x + 12 - 12x &\leq 2x^2 + 10x \\ \Leftrightarrow \quad x^2 - 7x + 12 &\leq 0 \\ \Leftrightarrow \quad (x-3)(x-4) &\leq 0 \end{aligned}$$

must hold, and this is only possible for $x = 3$ or $x = 4$. In both cases, there do exist polyhedra with the required property (Figure 13), and these are the only two. qed

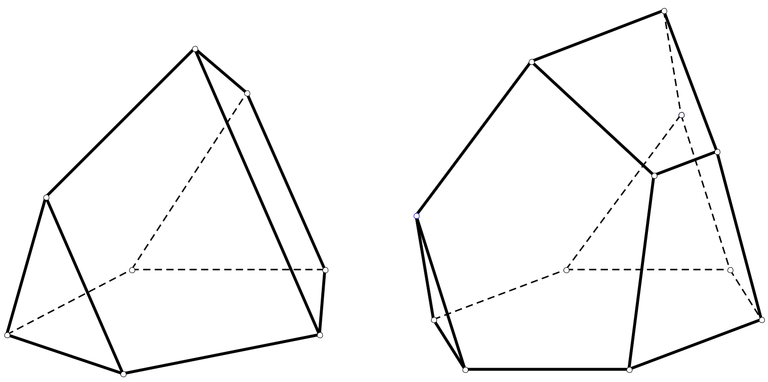


Figure 13

Note that the "uniqueness" of these polyhedra is meant in a vaguely topological sense, i.e. all kinds of transformations can yield different looking polyhedra, but these will have essentially the same structure with respect to the relative positions of vertices, edges and faces.

Solution to Problem 4: (The idea for this proof is due to Martin Windischer.) Again we let n_{max} denote the maximum number of sides of any of the faces of the polyhedron and $x = n_{max} - 2$. If f_k denotes the number of faces of P with k sides, we have

$$f = f_3 + f_4 + \ldots + f_{x+2},$$

with $f_i \in \{0, 1, 2\}$ for $3 \leq i \leq x+1$ and $f_{x+2} \in \{1, 2\}$.
We define the number of "missing faces"

$$t := 2x - f$$

and the number of "missing edges"

$$s := \sum_{i=3}^{x+2} (2 - f_i) \cdot i.$$

For the defining values of P, we then have

$$\begin{aligned} f &= 2x - t \\ e &= \frac{1}{2} \cdot (x(x+5) - s) \\ \text{and} \quad v &\leq \frac{1}{3} \cdot (x(x+5) - s). \end{aligned}$$

By Euler's formula, we therefore have

$$\frac{1}{2}\cdot(x^2+5x+4-s) = e+2 = f+v \leq \frac{1}{3}\cdot(x^2+5x-s)+2x-t$$

$$\Leftrightarrow 3x^2+15x+12-3s \leq 2x^2+10x-2s+12x-6t$$

$$\Leftrightarrow x^2-7x+12 \leq s-6t.$$

We now note that t and s are not completely independent. If $t=0$, we must have $s=0$. If $t=1$, we have $s \leq x+2$, since the number of edges of the single "missing face" is at most $x+2$. Similarly, we have $s \leq (x+2)+(x+1) = 2x+3$ for $t=2$, $s \leq (x+2)+2(x+1) = 3x+4$ for $t=3$, $s \leq (x+2)+2(x+1)+x = 4x+4$ for $x=4$, and so on. In general we have

$$s \leq tx + w(t),$$

where $w(t)$ is defined by $w(0)=0$, $w(1)=2$ and $w(t+2) = w(t)+3-t$.

In order for it to be possible that

$$x^2-7x+12 \leq s-6t$$

holds for some value of x, it must be true that

$$x^2-7x+12 \leq s-6t \leq tx+w(t)-6t$$

must hold. This is only possible if

$$x^2-7x+12-tx-w(t)+6t = 0$$

has a real root, i.e. if its discriminant is non-negative. This is the case if

$$\begin{aligned} & (t+7)^2-4\cdot(12-w(t)+6t) \geq 0 \\ \Leftrightarrow\ & t^2+14t+49-48+4w(t)-24t \geq 0 \\ \Leftrightarrow\ & t^2-10t+1+4w(t) \geq 0. \end{aligned}$$

This inequality is certainly correct for $t=0$ and $t=1$, since both

$$0^2-10\cdot 0+1+4\cdot 0 = 1 \geq 0$$

and

$$1^2-10\cdot 1+1+4\cdot 2 = 0 \geq 0$$

hold. We shall now show by induction that the inequality is not correct for any $t \geq 2$.

For $t=2$, we have

$$2^2-10\cdot 2+1+4\cdot 3 = -3 < 0$$

and for $t=3$

$$3^2-10\cdot 3+1+4\cdot 4 = -4 < 0.$$

If we assume that

$$t^2-10t+1+4w(t) < 0$$

holds for some t, we have

$$\begin{aligned} 4w(t) &< -t^2+10t-1 \\ \Rightarrow 4w(t) &< -t^2+10t-1+4 \\ \Rightarrow 4(w(t)+3-t) &< -t^2+10t-1+4+12-4t \\ &= -t^2-4t-4+10t+20-1 \\ &= -(t+2)^2+10(t+2)-1, \end{aligned}$$

or

$$(t+2)^2 - 10(t+2) + 1 + 4w(t+2) > 0,$$

and it follows that

$$t^2 - 10t + 1 + 4w(t) < 0$$

holds for all $t \geq 2$. No polyhedron with the required properties can therefore exist with $t \geq 2$. The only possible values for t are therefore 0 or 1. The case $t = 0$ was discussed in problem 27, and it only remains to consider the case $t = 1$.

For $t = 1$ we have $s \leq x + 2$, and therefore

$$\begin{aligned} & x^2 - 7x + 12 \leq s - 6t \leq x + 2 - 6 \\ \Leftrightarrow\ & x^2 - 8x + 16 \leq 0 \\ \Rightarrow\ & x \in \{1, 2, \ldots, 9\}. \end{aligned}$$

Since t is equal to 1 and P has a face with $x + 2$ sides, each of which has a common edge with a different face, we have $f \geq x + 3$, and due to $f = 2x - t$, this implies

$$2x - 1 \geq x + 3,$$

or $x \geq 4$. Since

$$s \leq tx + w(t) = x + 3$$

holds and s is even due to $e = \frac{1}{2} \cdot (x(x+5) - s)$ (noting that one of x and $x + 5$ must be even and that e is an integer), we must consider various possibilities. If $x = 4$, we have $s \leq 4 + 2 = 6$ on the one hand, and due to $s \geq x^2 - 7x + 12 + 6t$, $s \geq 16 - 28 + 12 + 6 = 6$ on the other hand. The only possible value for s in this case is therefore $s = 6$.

If $x = 5$, we have $s \leq 5 + 2 = 7$ on the one hand and $s \geq 25 - 35 + 12 + 6 = 8$ on the other hand, and there can be no possible value for s. Since similar contradictions are obtained for all larger values of x, we see that there is only one possible polyhedron P with $t = 1$. This polyhedron has $x = 4$, and therefore $f = 6$, and since $s = 6$, it must have two 3-sided faces, two 4-sided faces, two 5-sided faces and one 6-sided face. Such a polyhedron is shown in Figure 14. We see that there are only three possible polyhedra with the required property. qed

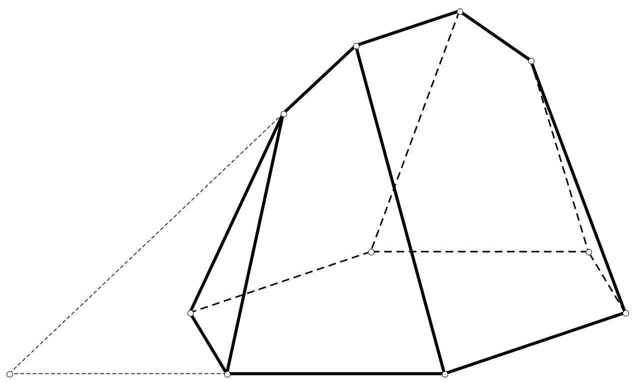


Figure 14

5 "n-faced" Polyhedra

In this section, we will consider problems pertaining to polyhedra whose faces all (or almost all) have the same number of sides. We name a convex polyhedron "n-faced", if its faces all have exactly n sides. For instance, deltahedra are examples of 3-faced polyhedra. If the faces of a convex polyhedron are all n-gons with the same value of n with the exception of a small, well-defined number of faces, with a different number of sides, we call the polyhedron "quasi n-faced".

Problem 28 *Prove that no n-faced polyhedron can exist for* $n \geq 6$.

Solution: If a polyhedron P has f faces, all of which have $n \geq 6$ sides, the fact that each edge of P is shared by two faces means that the number of edges e of P must fulfill the inequality

$$e = \frac{1}{2} \cdot n \cdot f \geq \frac{1}{2} \cdot 6f = 3f.$$

This, however, is a contradiction to the inequality

$$e \leq 3f - 6,$$

which was established in Problem 19. qed

Problem 29 *Prove that there cannot exist a 3-faced polyhedron with an odd number of faces.*

Solution: (This problem was posed as Problem 3 in [8].) If such a polyhedron exists with f triangular faces, the number of edges of the polyhedron in question must be

$$e = \frac{1}{2} \cdot 3f,$$

which is not an integer if f is odd, giving us a contradiction. qed

We note that this proof will also work if we wish to prove the impossibility of 5-faced polyhedra with an odd number of faces, or more generally of any polyhedron with an odd number of odd-sided faces.

Problem 30 *Let* P *be a 3-faced polyhedron with* f *faces. Determine all possible values of* f *for which such a polyhedron exists.*

Solution: In the preceding problem, we saw that f cannot be odd. Also, since P has a triangular face, which must have a common edge with each of three different faces, f cannot be smaller than 4. All even values for $f \geq 4$ are possible, however. For $f = 4$, P is simply a tetrahedron, and for $f = 2k$ with $k \geq 3$, a double k-sided pyramid has exactly $f = 2k$ triangular faces (Figure 15). qed

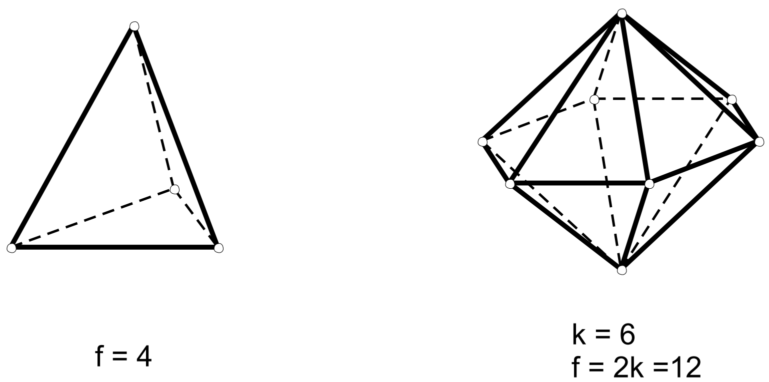


Figure 15

After considering these problems introducing us to the idea of n-faced polyhedra, we can turn our attention to the following series of questions dealing with the number of edges e of n-faced polyhedra.

Problem 31 *Let* P *be a 3-faced polyhedron with* e *edges. Prove* $3|e$.

Solution: P has f faces, and each of these faces has three sides. Each edge of P is common to two of the faces, and we therefore have

$$e = \frac{1}{2} \cdot 3f.$$

This number is certainly divisible by 3. qed

Since the proof is completely analogous, we also have the following:

Problem 32 *Let P be a 5-faced polyhedron with e edges. Prove* $5|e$.

Also, a quite similar problem is the following:

Problem 33 *Let P be a 4-faced polyhedron with an even number of sides and e edges. Prove* $4|e$.

Solution: Since P has an even number of faces, we can write $f = 2k$. The number of edges of P is therefore equal to

$$e = \frac{1}{2} \cdot 4f = \frac{1}{2} \cdot 4 \cdot 2k = 4k,$$

which is certainly divisible by 4. qed

If we take a closer look at the last 3 problems, it seems reasonable to ask the following question:

Problem 34 *Let P be an n-faced polyhedron with e edges. Is it always true that $n|e$ must hold?*

If we take a look at what we know so far, Problem 27 showed us that no n-faced polyhedron exists for $n \geq 6$, and it is quite obvious that there can be none for $n \leq 2$. For $n = 3$ and $n = 5$, we have just shown that the claim is true, as we have also shown for $n = 4$ if P has an even number of faces. Answering the question at hand is therefore closely related to answering the following question:

Problem 35 *Do 4-faced polyhedra with an odd number of faces exist?*

Solution to Problems 33 and 34: The surprising (to me) answer is that 4-faced polyhedra with an odd number of faces do indeed exist, as we can see in the examples due to Gottfried Perz and Christopher Albert in Figure 16.

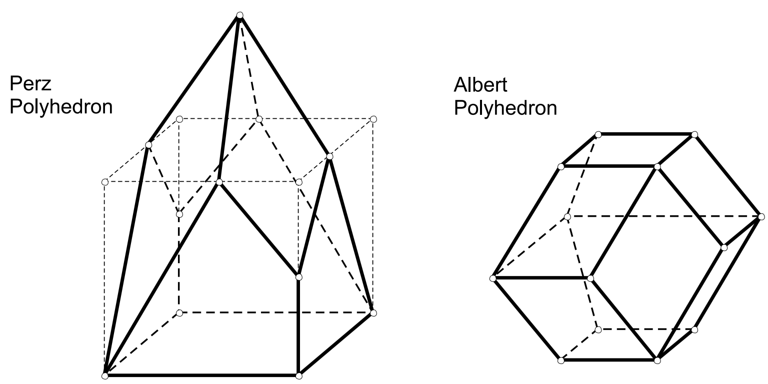


Figure 16

The "Perz Polyhedron" is derived from a cube, as can be seen in the Figure. The resulting polyhedron has 9 faces, 18 edges and 11 vertices. The "Albert Polyhedron" results from a double truncated quadratic pyramid as shown. This polyhedron has 11 faces, 22 edges and 13 vertices. In both cases, we see that the number of edges (18 and 22 respectively) is not divisible by the number of sides of each face (4). qed

Having determined the existence of such 4-sided polyhedra with odd values of f, it is interesting to ask the analogous question to Problem 30 for 4-sided polyhedra.

Problem 36 *Let P be a 4-faced polyhedron with f faces. Determine all possible values of f for which such a polyhedron exists.*

Solution: We shall show that 4-faced polyhedra with f faces exist for all values $f \geq 8$ and for $f = 6$, but not for $f = 7$ or $f \leq 5$. In order to prove this, we shall divide the proof into the following steps:

a) 4-faced polyhedra exist for all even values of $f \geq 6$.

b) 4-faced polyhedra exist for all odd values of $f \geq 9$.

c) No 4-faced polyhedra exist for $f \leq 5$.

d) No 4-faced polyhedra exist for $f = 7$.

ad a): If $f = 2k$ with $k \geq 3$, there certainly exists a 4-sided double "k-pyramid", as illustrated in Figure 17.

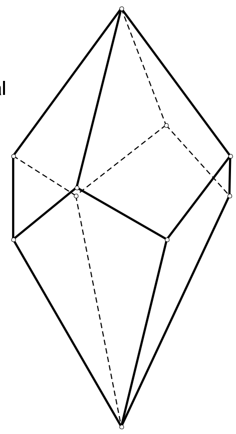


Figure 17

ad b): We already know of the existence of 4-sided polyhedra with 9 and 11 sides from the preceding problem. If we can show that the existence of a 4-sided polyhedron with f faces implies the existence of one with $f + 4$ faces, this part of the proof is finished.

This is indeed the case. If P is a 4-sided polyhedron with f faces , and the vertices of one such face are A, B, C and D, we can define a new polyhedron P' with $f + 4$ faces in the following manner. The planes of the faces of P divide space into a finite collection of sections, two of which border on the quadrilateral $ABCD$. P is one of these. We choose a point S in the interior of the other, and define A' as the mid-point of SA, B' as the mid-point of SB, and similarly C' and D'. We now define P' as having all faces in common with P except $ABCD$, which we replace by the five quadrilaterals $A'B'C'D'$, $ABB'A'$, $BCC'B'$, $CDD'C'$ and $DAA'D'$. If P was convex, so is P', and P' certainly has $f + 4$ 4-sided faces, as required.

ad c): If a polyhedron P has a 4-sided face, it must have at least four more faces, since each side of the 4-sided face must be a common edge with a different face. We therefore certainly have $f \geq 5$. If $f = 5$, P must have $e = \frac{4 \cdot 5}{2} = 10$ edges, and therefore $v = 10 + 2 - 5 = 7$ vertices. This is not possible, since P must have a face $ABCD$ with at least three edges through each of the vertices A, B, C and D. One each of these is not a side of $ABCD$, and none of these can have a common end-point, since this would mean that P would have a triangular face. (If the edges through A and B had a common vertex E, for instance, ABE would be a face of P.) We see that $f \leq 5$ is not possible for 4-sided polyhedra.

ad d): To prove the impossibility of $f = 7$ for a 4-sided polyhedron P, it is useful to have a look at what a Schlegel diagram of P would look like, if it existed. We assume that such a P does indeed exist. Since $f = 7$, we have $e = \frac{4 \cdot 7}{2} = 14$ and $v = 14 + 2 - 7 = 9$. If each of the vertices were trihedral, the number of faces would be $f = \frac{3 \cdot 9}{4}$, which is not possible. Indeed, since we know that $f = 7$, the only possibility for the 9 vertices is that 8 of them are trihedral and one tetrahedral, since this is the only case that yields $f = \frac{3 \cdot 8 + 4}{4} = 7$.

If A is the tetrahedral vertex, P has edges AB, AC, AD and AE. Since each of the faces of P is 4-sided, P must have faces $ABFC$, $ACGD$, $ADHE$ and $AEKB$. Defining these four faces, we have already "used up" all 9 vertices and 12 edges, and the vertices F, G, H and K can therefore not be trihedral, which is a contradiction. We see that $f = 7$ is not possible for 4-sided polyhedra. qed

Having asked which values of f are possible for 3- and 4-faced polyhedra, it is interesting to ask the same of 5-faced polyhedra. The only such polyhedron known to most people is the dodecahedron, and it takes some thought to find another. One question we can therefore ask is the following.

Problem 37 *We are given a polyhedron whose f faces are all pentagons. Can f have any value other than 12? Is there an upper limit to the possible values of f?*

Solution: There is no upper limit to the possible values of f. One way to see this is to note that it is possible to augment any 5-sided polyhedron with more pentagonal sides in a similar fashion to that described for 4-sided polyhedra in part b) of the preceding problem.

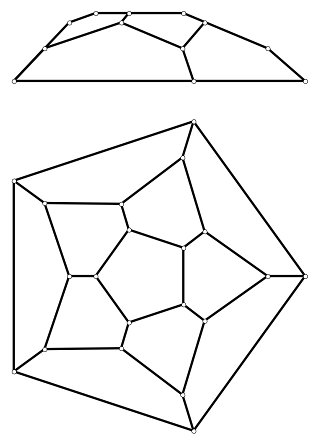


Figure 18

Figure 18 shows us the horizontal and vertical projections of such an augmentation. The horizontal projection is simultaneously the Schlegel diagram of a dodecahedron, and it is always possible to replace a pentagon in the Schlegel diagram of any 5-sided polyhedron with this graph. The full proof that such an augmentation is always possible is somewhat more convoluted, but application of the result on 3-connected graphs and polyhedra mentioned in section 2 yields one such proof. qed

Applying the augmentation described in this problem to the dodecahedron shows us that we can successively replace any pentagonal face of a 5-sided polyhedron by 11 such faces. This means that we can "build" 5-sided polyhedra with $f = 2 + 10k$ faces for any positive integer values of k. Of course, this immediately leads us to the next question, namely

Problem 38 *Does there exist a 5-sided polyhedron with f faces and $f \neq 2 + 10k$ for all positive integer values of k?*

Solution: The answer to this question is yes. An interesting class of such polyhedra is due to Gerd Baron, an example of which is shown in Figure 19.

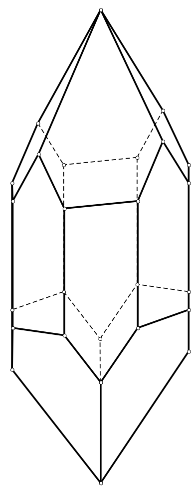


Figure 19

The middle part of such a polyhedron is made up of pentagons resulting from the rectangular sides of a regular $2k$-sided prism, augmented by points alternately attached to opposite sides. "Closing" the polyhedron with the planes joining these points with the sides of the pentagons as shown yields a polyhedron with $4k$ pentagonal sides for any value of $k \geq 3$. This means that we can obtain such values for f as $4 \cdot 4 = 16$ or $4 \cdot 5 = 20$, which are not of type $2 + 10k$. qed

Of course, we would like complete information concerning which values of f are possible for 5-sided polyhedra. We know from the comment after Problem 29 that f can never be odd. Since all values of the form $f = 2 + 10k$, $k \geq 1$ and $f = 4j$, $j \geq 3$ are possible, we can produce 5-sided polyhedra with

$$f = 12, 16, 20, 22, 24, 28, 32, \ldots.$$

The augmentation described in Problem 37 can also be applied to "Baron Polyhedra", of course, and so we can also produce 5-sided polyhedra with $f = 16 + 10k$, $f = 20 + 10k$, $f = 24 + 10k$ and $f = 28 + 10k$. This means that all even values of $f \geq 20$ are possible. Are $f = 14$ and $f = 18$ possible? I do not know the answer to this question yet. I beleive that $f = 14$ may not be possible, but $f = 18$ is unclear.

What remains to be shown is the following:

Problem 39 *Prove that there cannot exist a 5-sided polyhedron with f faces and $f < 12$.*

I leave the proof of this to the reader (mainly because I can't think of an elegant proof, even though the result seems so obvious).

6 "Two-faced" Polyhedra

There is no fundamental way for polyhedra to be dishonest or unfaithful, and their faces cannot be 2-sided. Obviously, something else is implied in the title of this section.

We define a "two-faced" polyhedron as a convex polyhedron P, whose faces are all either k-sided or m-sided, with k and m being different positive integers. In order for P to qualify as being two-faced, we assume that f_k and f_m are both greater than 0, whereby f_i denotes the number of faces with i sides. If $f_k = p$ and $f_m = q$, we will call P a "$(p_k; q_m)$-polyhedron".

Most commonly studied polyhedra fall into this category. Consider the following table:

$$\begin{aligned} \text{regular } n-\text{sided pyramid}, \ n>3: &\quad (n_3; 1_n) \\ \text{regular } n-\text{sided prism}, \ n\geq 3, \ n\neq 4: &\quad (n_4; 2_n) \\ \text{regular } n-\text{sided anti}-\text{prism}, \ n>3: &\quad (2n_3; 2_n) \\ \text{some Archimedean solids}: &\quad (12_5; 20_6), \ (8_3; 6_4), \ (4_3; 4_6). \end{aligned}$$

While it may seem that such polyhedra are too well known to make them well suited to this type of question, they are in fact very much so.

Problem 40 *Does there exist a $(1_3; q_4)$-polyhedron for some positive integer q? If so, determine the smallest possible value of q. If not, prove why not.*

Solution: There can be no such polyhedron. If one existed, it would have

$$e = \frac{1\cdot 3 + q\cdot 4}{2} = 2q + 1 + \frac{1}{2}$$

edges, and this is not a whole number. qed

Problem 41 *Does there exist a $(1_k; q_m)$-polyhedron with an even value of k and an odd value of q?*

Solution: The only "standard" $(1_k; q_m)$ polyhedron is the n-sided pyramid, for which we have $k = q = n$ (and $m = 3$). The parity of k and q must be the same for any pyramid, of course. If a polyhedron with the required property exists, it cannot be a pyramid.

Since the number of edges of such a polyhedron is

$$e = \frac{k + q\cdot m}{2} = \frac{k}{2} + q\cdot\frac{m}{2},$$

m must be an even number. If both k and m are to be even (but different), it is not a good idea to search for solutions where $m > 4$ (see also Problem 42). An obvious place to start looking for a polyhedron with the required property is $k = 6$ and $m = 4$. There does indeed exist a $(1_k; q_m)$ polyhedron with $k = 6$, $m = 4$ and q odd, for instance the $(1_6; 9_4)$-polyhedron as shown in Figure 20. qed

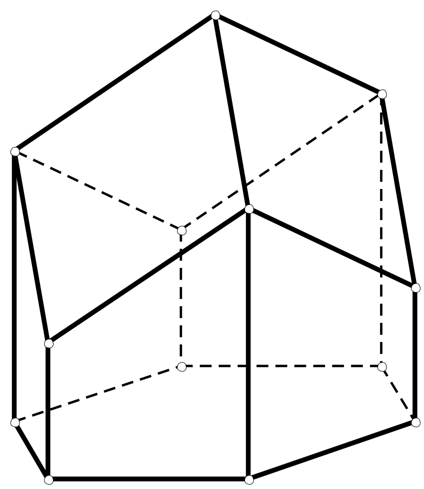


Figure 20

Problem 42 *Prove that there cannot exist a $(1_k; q_m)$-polyhderon with $m \geq 6$.*

Solution: If such a polyhedron did exist, it would have $f = q+1$ faces and $e = \frac{m}{2} \cdot q + \frac{k}{2}$ edges. In Problem 19 we saw that

$$e + 6 \leq 3f$$

must hold for any convex polyhedron, and it would therefore follow that

$$\frac{m}{2} \cdot q + \frac{k}{2} + 6 \leq 3q + 3 \quad \Leftrightarrow \quad \left(\frac{m}{2} - 3\right) \cdot q + \frac{k}{2} + 3 \leq 0$$

must hold. If $m \geq 6$ holds, it follows that $\frac{m}{2} - 3 \geq 0$ also holds however, and this is then certainly not possible. qed

Problem 43 *Prove that there cannot exist a* $(1_4; q_m)$*-polyhedron with even* m*.*

Solution: If m is even and not equal to 4, we must have $m \geq 6$, since each face of a polyhedron must have at least 3 sides. The result of the preceding problem therefore shows us that such a polyhedron cannot exist. qed

Problem 44 *Does there exist a* $(1_k; q_m)$*-polyhedron with* $k < m$*?*

Solution: If such a polyhedron exists, there are many limitations to the possible values of k, m and q. First of all, we can not have $k \geq 5$ (and therefore $m \geq 6$), as was just shown in Problem 42. The impossibility of a $(1_3; q_4)$-polyhedron was shown in Problem 40. The only categories of such polyhedra that can possibly exist are therefore $(1_3; q_5)$ and $(1_4; q_5)$. In order for the number of edges to be an integer, the value of q must be odd in the former case, and even in the latter. Finding a polyhedron in either case is not easy. One example for a $(1_3; q_5)$-polyhedron is shown in Figure 21. (I do not know yet whether a $(1_4; q_5)$-polyhedron exists or not.)

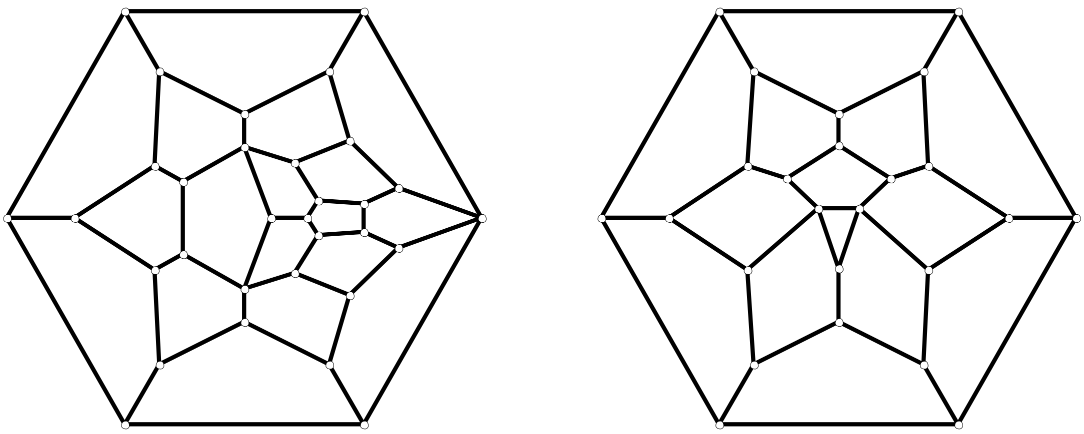


Figure 21

In this figure, we see the polyhedron cut in two. Each piece has a hexagonal periphery, and the Schlegel diagrams of each half are illustrated. One piece is made up of only 18 pentagons, and the other is made up of 13 pentagons and one triangle. The resulting polyhedron is therefore of type $(1_3; 31_5)$. qed

7 Additional Questions

In this final section, I have listed a number of additional questions suggested by the problems discussed in this paper. I have not to ordered them according to difficulty. While some of these problems are quite simple, some are not at all easy to solve. Those for which I do not know the full answer are marked with an asterisk. I hope that readers will find some enjoyment in solving these problems.

Problem 45 *Does a 4-faced polyhedron exist, such that an even number of faces meet at each vertex? If so, what is the smallest number of faces that such a polyhedron can have?*

Problem 46 *Is it true that the edges of a 4-faced polyhedron can always be colored with two colors, such that no two edges of the same color meet in a common vertex?*

Problem 47 *Does a 4-faced polyhedron exist, such that exactly three edges meet in each vertex and $f > 6$? For which values of f does a polyhedron with this property exist? (Note that a cube is such a polyhedron with $f = 6$.)*

Problem 48 * *Let v (the number of vertices of a polyhedron) be given. Determine the smallest possible values for f and e.*

Problem 49 * *For which values of p can we find k and m such that a $(p_k; p_m)$-polyhedron exists?*

Problem 50 * *For which values of q can we find k and m such that a $(1_k; q_m)$-polyhedron exists?*

Problem 51 *For which values of q does a $(1_4, q_3)$-polyhedron exist?*

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