Folding the Regular Nonagon

Robert Geretschläger, Bundesrealgymnasium, Graz, Austria

Introduction

In the March 1997 of *Crux* [1997: 81], I presented a theoretically precise method of folding a regular heptagon from a square of paper using origami methods in an article titled *"Folding the Regular Heptagon"*. The method was derived from results established in *"Euclidean Constructions and the Geometry of Origami"* [2], where it is shown that all geometric problems that can be reduced algebraically to cubic equations can be solved by elementary methods of origami. Specifically, the corners of the regular heptagon were thought of as the solutions of the equation

$$z^7 - 1 = 0$$

in the complex plane, and this equation was then found to lead to the cubic equation

$$\zeta^3 + \zeta^2 - 2\zeta - 1 = 0,$$

which was then discussed using methods of origami. Finally, a concrete method of folding the regular heptagon was presented, as derived from this discussion.

In this article, I present a precise method of folding the regular nonagon from a square of paper, again as derived from results established in [2]. However, as we shall see, the sequence of foldings used is quite different from that used for the regular heptagon. As for the heptagon, the folding method is once again presented in standard origami notation, and the mathematical section cross-referenced to the appropriate diagrams.

Angle Trisection

For any regular n-gon, the angle under which each side appears as seen from the mid-point is $\frac{2\pi}{n}$. Specifically, for $n = 9$, the sides of a regular nonagon are seen from its mid-point under the angle $\frac{2\pi}{9}$. As is well known, this angle cannot be constructed by Euclidean methods. Three times this angle (or $\frac{2\pi}{3}$) can, however, and we note that it would be possible to construct a regular nonagon by Euclidean methods, if it were possible to trisect an arbitrary angle, or at least the specific angle $\frac{2\pi}{3}$. If this were the case, all that we would have to do would be to construct an equilateral triangle, trisect the angles from its mid-point to its corners, and intersect these with the triangle's circumcircle. Unfortunately however, as generations of mathematicians have been forced to accept (although there are a few hold-outs still out there), angle trisection by Euclidean methods of straight-edge and compass is impossible, as is the construction of a regular nonagon.

The underlying reason for the impossibility of angle trisection by Euclidean methods is the fact that straight-edge and compass constructions only allow the solution of problems that reduce algebraically to linear or quadratic equations. Angle trisection, however, involves the irreducible cubic equation

$$x^3 - \frac{3}{4}x - \frac{1}{4}\cos 3\alpha = 0,$$

which derives from the well established fact (see for instance [1]) that

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

For the specific case at hand, where $3\alpha = \frac{2\pi}{3}$, the cubic equation in question is

$$x^3 - \frac{3}{4}x + \frac{1}{8} = 0.$$

As shown in [2], the solutions of this equation are the slopes of the common tangents of the parabolas p_1 and p_2, whereby p_1 is defined by its focus

$$F_1\left(\tfrac{1}{16}, -\tfrac{3}{8}\right)$$

and its directrix

$$\ell_1 : x = -\frac{1}{16},$$

and p_2 is defined by its focus

$$F_2\left(0, \tfrac{1}{2}\right)$$

and its directrix

$$\ell_2 : y = -\frac{1}{2}.$$

(It is not too difficult to prove that this is indeed the case. Interested readers may like to try their hand at doing the necessary calculations themselves.) Finding the common tangents of two parabolas defined by their foci and directrices is quite straight-forward in origami, as it merely means making one fold, which places two specific points (the foci) onto two specific lines (the corresponding directrices). By this method, we will therefore now show how to fold a regular nonagon.

A Step-by-step Description of the Folding Process

As is usually the case in origami, we assume a square of paper to be given. We consider the edge-to-edge folds in step 1 as the x– and y–axes of a system of cartesian coordinates, and the edge-length of the given square as two units. The mid-point of the square is then the origin $M(0,0)$, and the end-points of the folds have the coordinates $(-1,0)$ and $(1,0)$, and $(0,-1)$ and $(0,1)$ respectively.

Steps 1 through 8 yield the foci and directrices of the parabolas discussed at the end of the second section of this article. The point C is the focus F_1 of parabola p_1, B is the focus F_2 of parabola p_2, and the creases onto which these two points are folded in step 9 are the directrices ℓ_1 and ℓ_2. Since the coordinates involved are all arrived at by halving certain line segments, it is quite easy to see that this is indeed the case.

The fold made in step 9 is then a common tangent of the parabolas, and its slope is therefore $\cos\frac{2\pi}{9}$. (This step, by the way, is the only one that cannot be replaced by Euclidean constructions.) The point of steps 10 to 13 is then to find the horizontal line represented by the equation $y = -\cos\frac{2\pi}{9}$. This is the horizontal fold through point E, as the distance between the vertical folds through points D and E is equal to 1.

We then obtain corners 2 and 9 of the nonagon (assuming the point with coordinates $(0, -1)$ to be corner 1) on this horizontal line by folding the unit length onto this line from mid-point M in step 14. Step 15 therefore yields the first two sides of the nonagon, and steps 16 and 17 complete the fold, making use of both the radial symmetry of the figure, and its axial symmetry with respect to the vertical line $M1$. Step 18, finally, shows us the completed regular nonagon.

The Folding Process

The diagrams follow the bibliography.

Conclusion

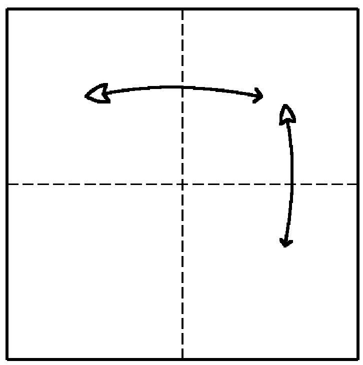
In the conclusion of *Folding the Regular Heptagon*, I declared myself as an ardent Heptagonist. I have no qualms or reservations about declaring myself an equally ardent Nonagonist now. Perhaps I will find some similar-minded folk out there willing to join me in my quest of popularizing these heretofore sadly neglected polygons.

References

[1] B. Bold, *Famous Problems of Geometry and How to Solve Them*, Dover Publications, Inc., Mineola, NY (1969).

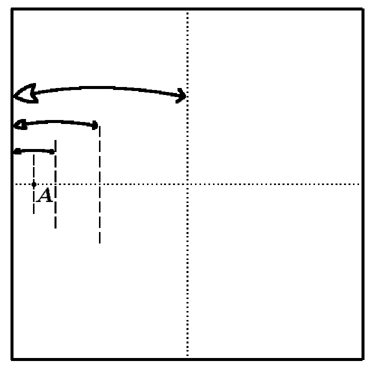
[2] R. Geretschläger, *Euclidean Constructions and the Geometry of Origami*, Mathematics Magazine, Vol. 68, No. 5 December 1995.

1.



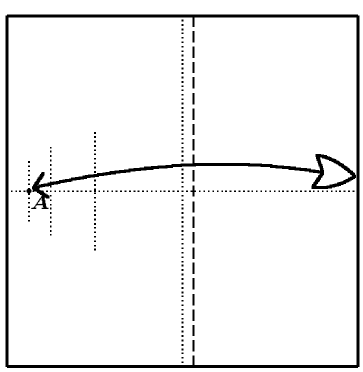
Fold and unfold twice.

2.



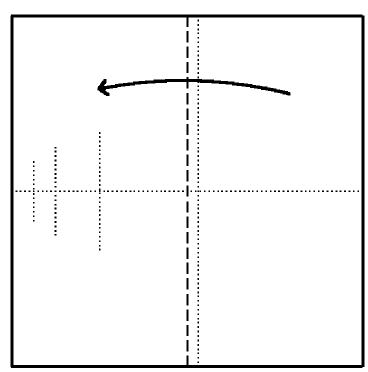
Fold and unfold three times,
making crease marks each time.
final crease yields point ***A***.

3.



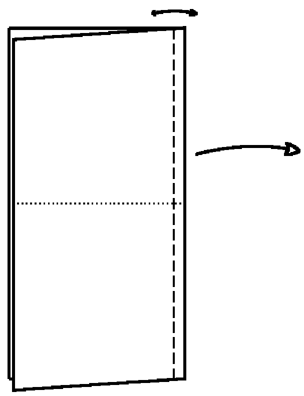
Fold edge to point ***A*** and unfold.

4.



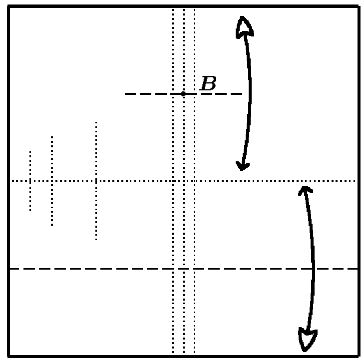
Refold edge to edge.

5.



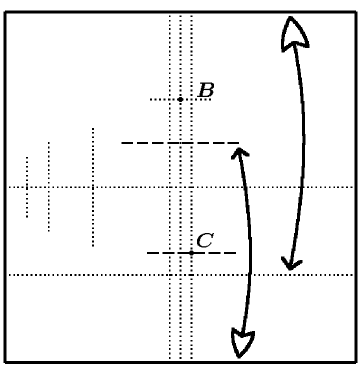
Fold and unfold both layers at crease, then unfold.

6.



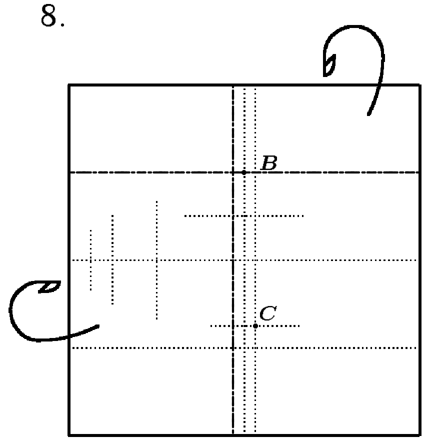
Fold and unfold twice, making crease mark at point B.

7.



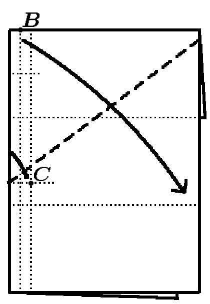
Fold upper edge to crease, unfold, then fold lower edge to new crease, making crease mark at point C.

8.



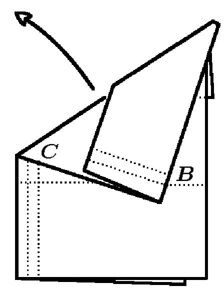
Mountain fold along creases.

9.



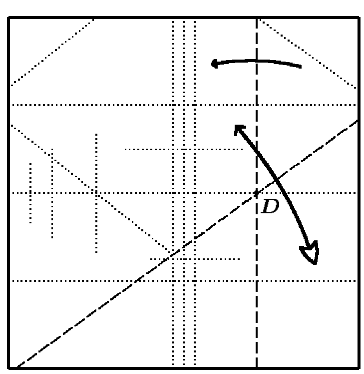
Fold so that ***B*** comes to lie on crease, and previous fold on ***C***.

10.



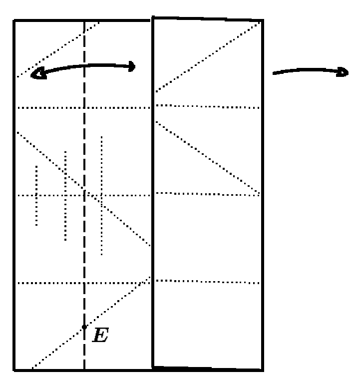
Unfold everything.

11.



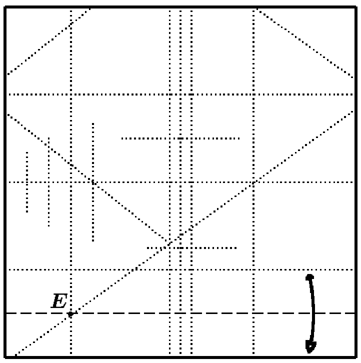
Fold along crease and unfold, then fold vertically through point ***D***.

12.



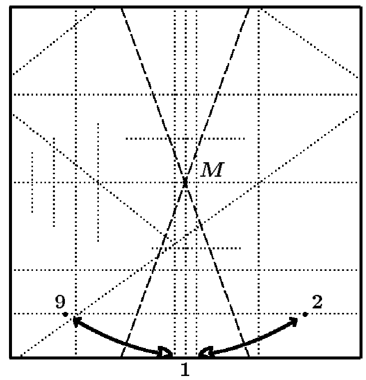
Fold edge to edge and unfold everything.

13.



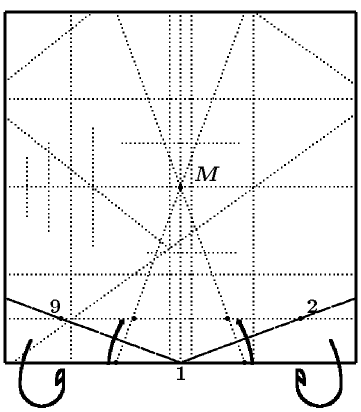
Fold horizontally through point E and unfold.

14.



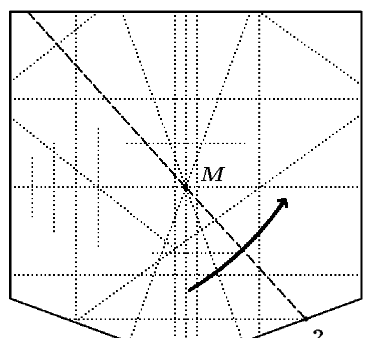
Fold through M such that point 1 lies on crease, unfold a repeat on other side (points 1, 2, 9 are corners of the nonagon).

15.



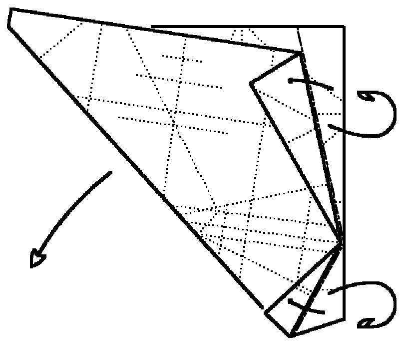
Fold back twice such that marked points come to lie on each other, resulting folds are sides of the nonagon.

16.



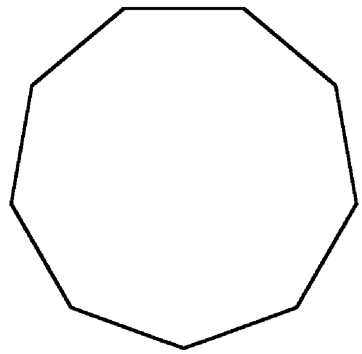
Fold through M and 2.

17.



Mountain fold lower layer using edges of upper layer as guide lines. Resulting folds are two more sides of the nonagon. Open up fold from step 16 and repeat steps 16 and 17 on left side, then fold through M and $\mathbf{2}$ once more. New folds are new guide lines. Repeating process completes the nonagon.

18.



The finished nonagon.