Folding Questions – A Paper about Problems about Paper

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INTRODUCTION

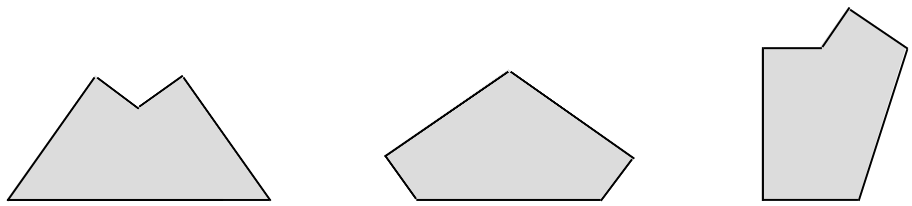
For many years now, I have been involved in the study of the geometry of paper folding. It therefore seems like an obvious step for me to have a closer look at problems in this area that have been used as competition questions in various countries. Although the result can hardly be called complete, what follows seems to be a fair representation of the kinds of ideas that have been used in this way.
In collecting these problems, I checked through various regional, national and international competitions from all around the world. I have intentionally focused on flat-folding problems, and have therefore not collected the many problems concerning nets of polyhedra or developments of cones or cylinders that can be found in the literature.
The full-length version of this paper is available on-line at http://geretschlaeger.brgkepler.at

1. SHAPES

The most elementary paper folding problems are those concerning the shape (i.e. the outline) of a flat folded piece of paper. The medium to be folded is almost always a rectangle or a square. In most such problems, the paper is folded over only once.

1) Three shapes X, Y and Z are shown below. A sheet of A4 paper (297 mm by 210 mm) is folded once and placed flat on a table. Which of these shapes could be made?

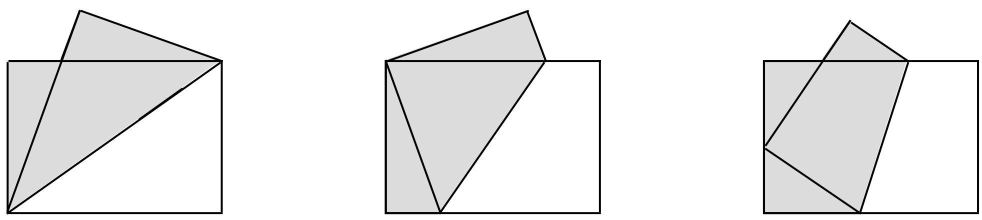
X Y Z



A) Y and Z only   B) Z and X only   C) X and Y only   D) none of them   E) all of them

source: UK Junior Mathematical Challenge 1999, Nr. 19.

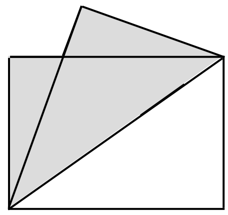
Solution: As we see in the figure, all are possible. The correct answer is therefore E.



2) A sheet of A4 size paper (297 mm x 210 mm) is folded once and then laid flat on the table. Which of these shapes could not be made?

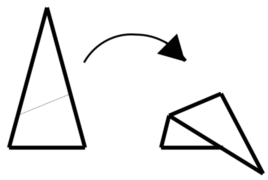
A) B) C) D) E)

source: UK Intermediate Mathematical Challenge 1999, Nr. 2.



Solution: As we can see in the figure, folding the rectangle along the diagonal does not yield the triangle shown in B. All other shapes are possible, and the correct answer is therefore B.

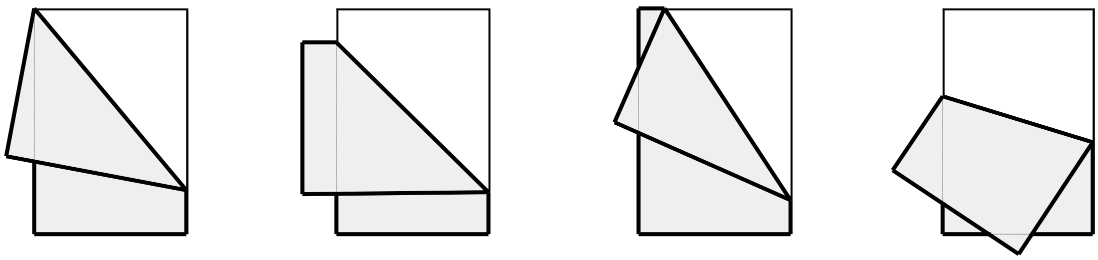
3) The diagram shows a triangular piece of paper that has been folded over to produce a shape with the outline of a pentagon. If a rectangular piece of paper is folded once, what is the smallest value of *n* (greater than 4) for which it is not possible to create an *n*-sided polygon in the same way?



A) 6 B) 7 C) 8 D) 9 E) 10

source: UK Senior Mathematical Challenge 2003, Nr. 10.

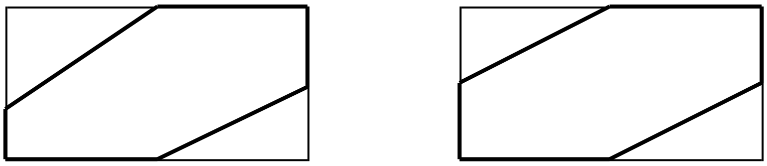
Solution: When the piece of paper is folded, the crease makes up one side of the resulting polygon. In addition, each of the four corners of the rectangle can contribute at most two sides to the resulting polygon. More than 9 sides are therefore certainly not possible. As we see in the figure, the other given values are possible.



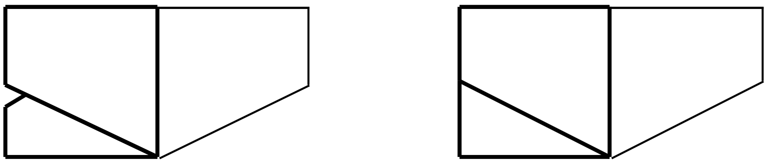
The correct answer is therefore E.

Comments

These problems are of a highly elementary type. An obvious idea for similar problems would be to consider a shape for the paper other than a rectangle. Also, any problem asking what is possible after two folds or more can become quite difficult to answer readily.
Another interesting variation results from turning the situation around, and asking which shapes can be folded to make a certain shape. We can, for instance ask, which of the following shapes can be folded once in order to form a square.



It may not be immediately obvious that it is possible for the hexagon on the right, but not for the one on the left, as we see below.

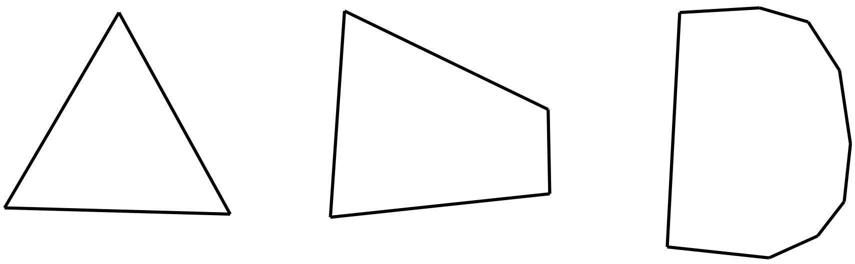


The difference is only in the placement of the end points of the sides of the hexagons on the left and right sides of the double squares. Similar variations are possible for the points on the top and bottom, of course.

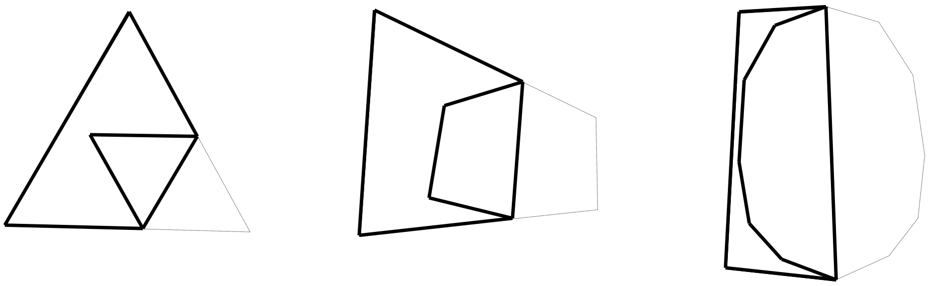
Some competition problems deal with *n*-gons that are folded to yield *m*-gons. This idea can be adapted in many ways. For instance, we can ask which values of m are possible if a specific value of n is given, or vice versa. In either case, we can either specify that the polygons are convex, or not. For specific values (n, m = 3, 4, …) this yields numerous questions, with some surprising answers. Let us consider one specific example.

Question: A convex *n*-gon is folded once. The result is a convex quadrilateral. Which values of *n* are possible?

Answer: All values $n \geq 3$ are possible. Consider the following figure:



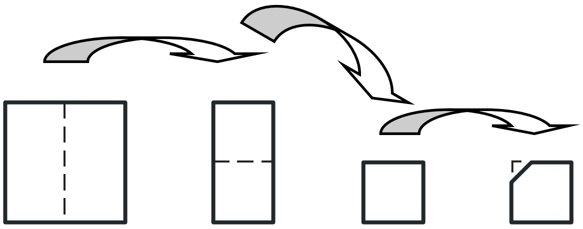
We are given polygons with $n = 3$, $n = 4$, $n = 9$, the last of which is exemplary for any $n > 4$. Each can be folded to produce a quadrilateral, as we see below.



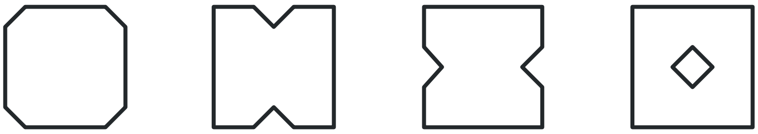
2. FOLD AND CUT

A small step leads us from considering the shapes that result from folding to shapes that result from folding, cutting, and unfolding again. One such problem is the following.

4) A square piece of paper is folded twice as shown in the figure, so that a small square results. A corner of the resulting square is cut off and the square is unfolded again. Which of the following shapes cannot result is this way?



A) B) C) D)



E) All shapes are possible.

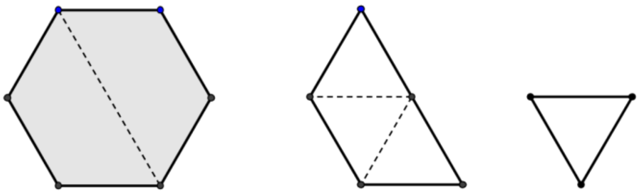
source: Kangaroo 2007, Écolier

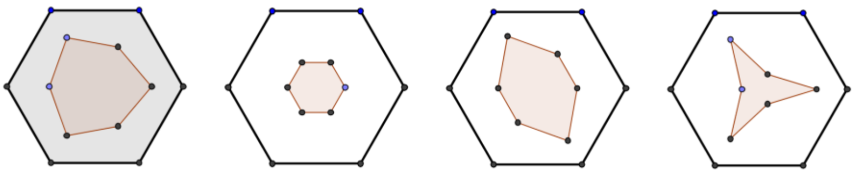
Solution: All four shapes are possible. In fact, they correspond to the four corners of the small square being cut. The correct answer is therefore E.

Comments

Problems of this type involve a bit of spatial reasoning, and there is always a certain aspect of powers of 2 involved, since any fold always doubles the number of layers of paper being considered. In essence, we are often considering simple versions of "snowflakes", i.e. the shapes created by folding a piece of paper multiple times in such a way that a center of symmetry is created (together with the lines of symmetry resulting from the folds), making some cuts, and then unfolding again, yielding a symmetric shape. It seems like an obvious variation to consider the hexagonal shape of a real snowflake. One such question could be the following:

Question: A piece of paper in the shape of a regular hexagon is folded over once and then into thirds as shown.
One straight cut is made, removing a section of the folded paper, and the remaining piece is unfolded again. Which of these shapes can be the result?





1 2 3 4

A) All are possible.
B) All except number 1 are possible.
C) All except number 2 are possible.
D) All except number 3 are possible.
E) All except number 4 are possible.

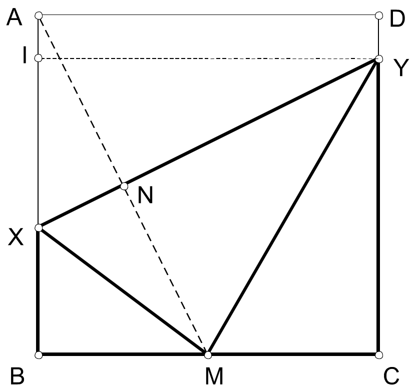
Answer: Because of the symmetry resulting from the folds, all but number 3 are possible. The correct answer is therefore D.

3. LENGTHS and AREA

The most important aspect in most elementary questions concerning measurable attributes of flat folding, and certainly in all that involve only a single fold, is that of the resulting symmetry. „Folding over" always implies leaving part of the paper stable on the folding surface, while reflecting the other part with respect to the folding edge.
Because of the resulting symmetry, the measures of angles and of lengths are retained, and this retention allows us to calculate a number of things that may, at first glance, appear not to be uniquely determined. This aspect is key to solving such problems.
An important aspect in many problems concerning lengths specifically, is the use of the Pythagorean Theorem. Right angles turn up in origami whenever a line is folded onto itself, for instance. They are therefore quite common, and calculation of lengths can therefore often be based on the sides of right triangles. Another important tool is the use of similar triangles, which also result quite naturally in folding problems because of the equal angles generated by the symmetries involved.

5) A square piece of paper *ABCD* is folded such that the corner *A* comes to lie on the mid-point *M* of the side *BC*. The resulting crease intersects *AB* in *X* and *CD* in *Y*. Show that $|AX| = 5\cdot |DY|$.

source: Mathematical Duel Bílovec – Chorzów – Graz – Přerov 2010



Solution: As shown in the figure, let *N* be the point of intersection of *AM* and the crease line *XY*. Furthermore, let the lengths of the sides of *ABCD* be equal to 1.

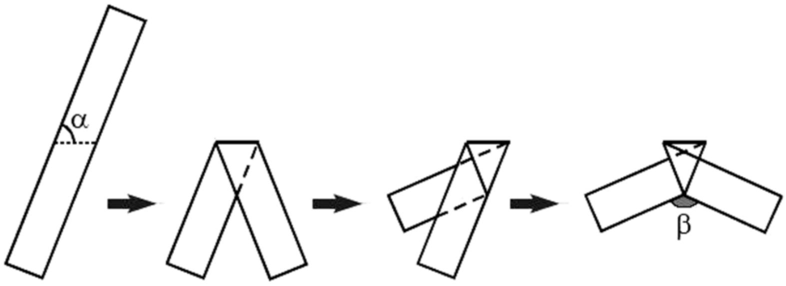
Since $|BM| = \frac{1}{2}$, we have $|AM| = \frac{\sqrt{5}}{2}$, and therefore $|AN| = \frac{1}{2}\cdot |AM| = \frac{\sqrt{5}}{4}$. The right-angled triangles *ABM* and *AXN* share the angle in *A*, and are therefore similar, and we have

$$|AX|:|AN| = |AM|:|AB| \Leftrightarrow |AX|:\frac{\sqrt{5}}{4} = \frac{\sqrt{5}}{2}:1,$$

and therefore $|AX| = \frac{5}{8}$. If *I* is the foot of *Y* on *AB*, the triangles *YIX* and *ABM* are congruent, since their sides are pairwise orthogonal, and $|AB| = |YI| = 1$ holds. We therefore have $|IX| = |BM| = \frac{1}{2}$, and therefore $|DY| = |AX|-|IX| = \frac{5}{8}-\frac{1}{2} = \frac{1}{8}$, and we see that $|AX| = 5|DY|$ holds as claimed.

4. ANGLES

6) A paper strip is folded three times as shown. Determine β if we are given that $\alpha = 70°$ holds.



A) 140° B) 130° C) 120° D) 110° E) 100°

source: Kangaroo 2010, Étudiant

Solution: Since the angle 70° is folded down, the small triangle in the middle of the strip in the second figure is isosceles with angles of 70°, 70° and 40°. Folding to the left and right therefore yields angles of 140° to the left and right, which have an angle of 40° in common. the angle in question is therefore equal to 360°-(140°+140°-40°) = 120°. The correct answer is therefore C.

7) Line *r* passes through the corner A of a sheet of paper and makes an angle α with the horizontal border, as shown in Figure 1. In order to divide α into three equal parts, we proceed as follows:

a) initially we mark two points B and C on the vertical border such that AB = BC; through B we draw a line *s* parallel to the border (Figure 2);

b) after that, we fold the sheet so as to make C coincide with a point C' on the line *r* and A with a point A' on line *s* (Figure 3); we call B' the point which coincides with B.

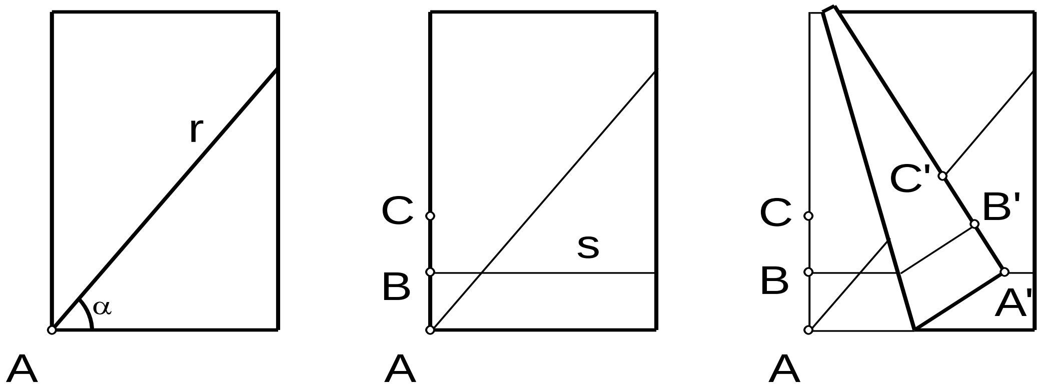
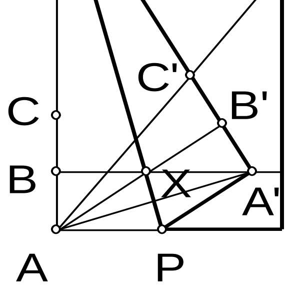


Figure 1 Figure 2 Figure 3

Show that lines AA' and AB' divide angle α into three equal parts.

source: 22nd Brazilian Mathematical Olympiad 2000, Nr. 1



Solution: Let P be the point in which the crease intersects the bottom edge of the paper and X the point in which the crease intersects with *s*. Furthermore, let β = ∠ PAA'. Since AP = AP', the triangle APA' is isosceles and we have ∠ PA'A = ∠ AA'P = β, and therefore ∠ PA'X = 2β, since AP and s are parallel. Since the triangles PAX and PA'X are congruent, we therefore also have ∠ PAX = 2β, and therefore ∠ XAA' = ∠ PAA' = β. Again, since AP and s are parallel, we therefore also have ∠ AXB = 2β, and thus ∠

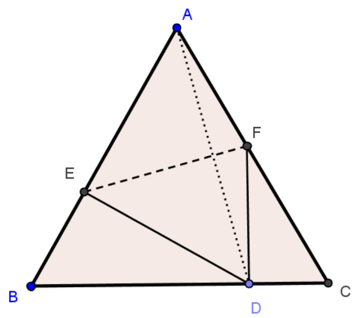
A'XB' = 2β. Since AP and s are parallel, B' therefore lies on the extension of AX. We see that AB' is both altitude and median in the triangle AA'C'. It is an altitude since A'B is perpendicular to AC, and therefore AB' must be perpendicular to A'C', and it is a median because B is the mid-point of AC, and B' must therefore be the mid-point of A'C'. We see that AA'C' must be isosceles, and AB' must also be the angle bisector in A, from which we deduce ∠ B'AC' = ∠ B'AA' = ∠ PAA' = β, proving the claim.
Note that the angle trisection described here is well established in the origami math literature, and is due to H. Abe.

Comments

It is not difficult to find variations on the themes of these problems. Calculating the lengths of line segments, the angles between line segments and the areas of triangles or quadrilaterals is fairly standard stuff, and not usually interesting enough to be the kind of stuff we are looking for in a competition problem, however.
One possible idea for an interesting situation may be the following.

Question: We are given an equilateral triangle *ABC* with sides of unit length. The point *A* is folded to the point *D* on *BC* as shown, resulting in the crease *EF* with *E* on *AB* and *F* on *AC*. We assume that *FD* is perpendicular to *BC*.

a) Determine the angle ∠ *AED*.
b) Determine the length of the line segment *CD*.
c) Determine the ratio of the areas of the triangles *AEF* and *ABC*.



Answer:
a) ∠ *AED* = 90°
b) |*CD*| = 2 - √3
c) [*AEF*]:[*ABC*] = (3√3 – 5):1