Folding Questions – A Paper about Problems about Paper

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Presented at WFNMC-6, Riga, Latvia, July 27th, 2010

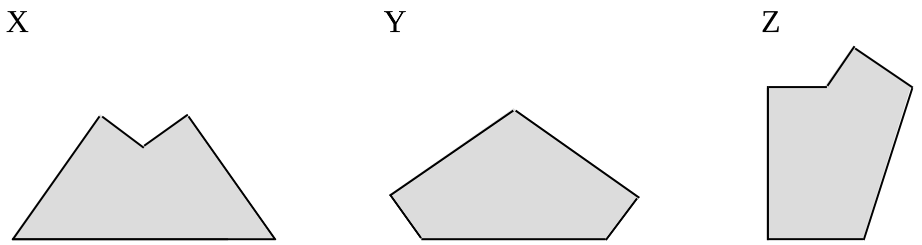
INTRODUCTION

For many years now, I have been involved in the study of the geometry of paper folding. It therefore seems like an obvious step for me to have a closer look at problems in this area that have been used as competition questions in various countries. Although the result can hardly be called complete, what follows seems to be a fair representation of the kinds of ideas that have been used in this way.
In collecting these problems, I checked through various regional, national and international competitions from all around the world. I have intentionally focused on flat-folding problems, and have therefore not collected the many problems concerning nets of polyhedra or developments of cones or cylinders that can be found in the literature. Such a collection could make for an interesting paper at some future point in time.
Most of the problems in this collection were set at lower levels, but many are not at all as easy to solve as one might expect, and some are even mathematically quite interesting. The problems are presented arranged by sub-topic, and together with the overview of problems already to be found in the literature, for some of the sub-topics I also present some ideas for related potential competition problems.

1. SHAPES

The most elementary paper folding problems are those concerning the shape (i.e. the outline) of a flat folded piece of paper. The medium to be folded is almost always a rectangle or a square, as these are the standard shapes in which paper is generally available. In most such problems, the paper is folded over only once. Sometimes the problems are embedded in quite innovative settings.

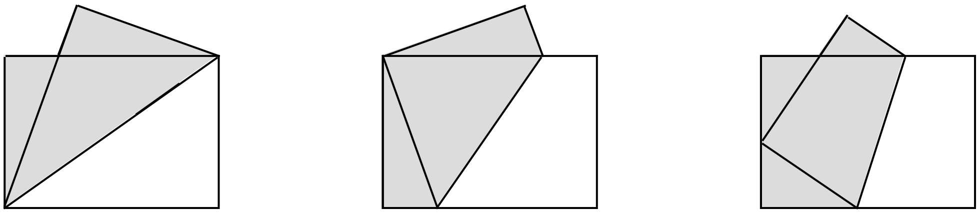
1) Three shapes X, Y and Z are shown below. A sheet of A4 paper (297 mm by 210 mm) is folded once and placed flat on a table. Which of these shapes could be made?



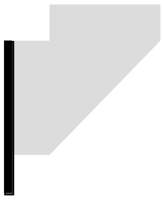
A) Y and Z only   B) Z and X only   C) X and Y only   D) none of them   E) all of them

source: UK Junior Mathematical Challenge 1999, Nr. 19.

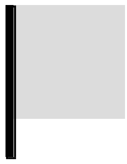
Solution: As we see in the figure, all are possible. The correct answer is therefore E.

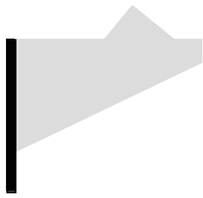


2) Carla is looking out her window. She sees a rectangular flag moving in the wind. She copies the outline of the flag as it appears to her five times. Which of the following pictures cannot be correct if we know that the flag was not torn?

A) 

B) 

C) 

D) 

E) 

source: Kangaroo 2020, Écolier

Solution: Since the flag is not torn, the side connecting it to the flagpole can never appear shorter than it actually is. The correct answer is therefore B.

3) A sheet of A4 size paper (297 mm x 210 mm) is folded once and then laid flat on the table. Which of these shapes could not be made?

A)

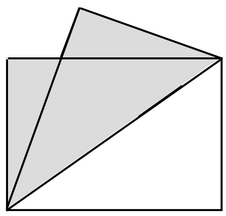
B)

C)

D)

E)

source: UK Intermediate Mathematical Challenge 1999, Nr. 2.



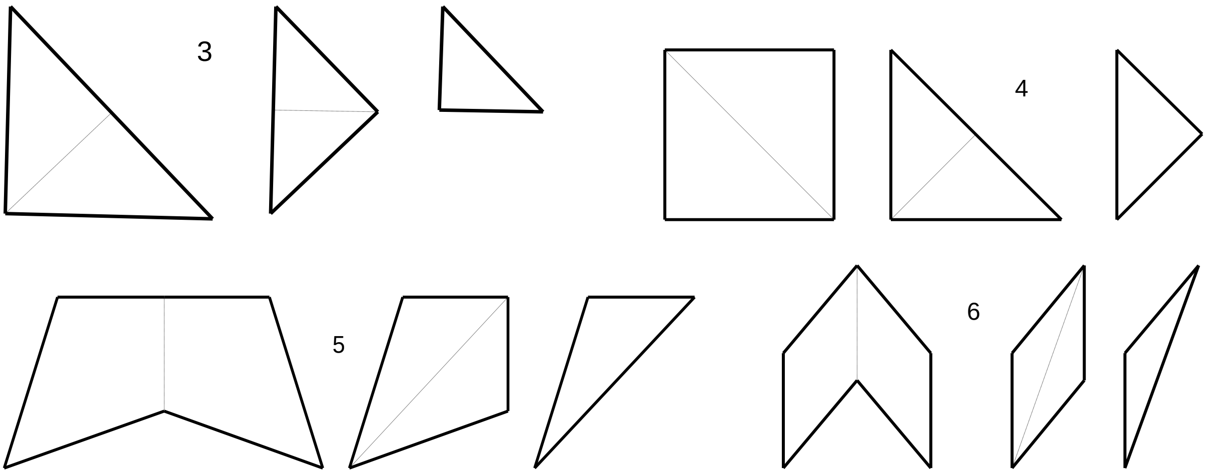
Solution: As we can see in the figure, folding the rectangle along the diagonal does not yield the triangle shown in B. All other shapes are possible, and the correct answer is therefore B.

4) A piece of paper in the shape of a polygon is folded in half along a line of symmetry. The resulting shape is also folded in half, again along a line of symmetry. The final shape is a triangle. How many possibilities are there for the number of sides of the original polygon?

A) 3 B) 4 C) 5 D) 6 E) 7

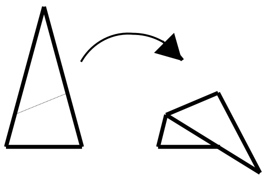
source: UK Junior Mathematical Challenge 2007, Nr. 25.

Solution: If we consider unfolding the final triangle, we see that the edge of this triangle that was the crease is in the interior of the previous polygon. The other two edges of the final triangle have mirror images in the previous polygon, which can be extensions of the sides of the triangle. The intermediate polygon can therefore have at most 4 sides. Similarly, the original polygon can have at most 6 sides. (In general, unfolding an *n*-gon yields an *m*-gon with $m \leq 2(n-1)$.) The possibilities for the number of sides of the original polygon are therefore 3, 4, 5 and 6. As we see in the following figure, all of these cases are indeed possible.



The correct answer is therefore B.

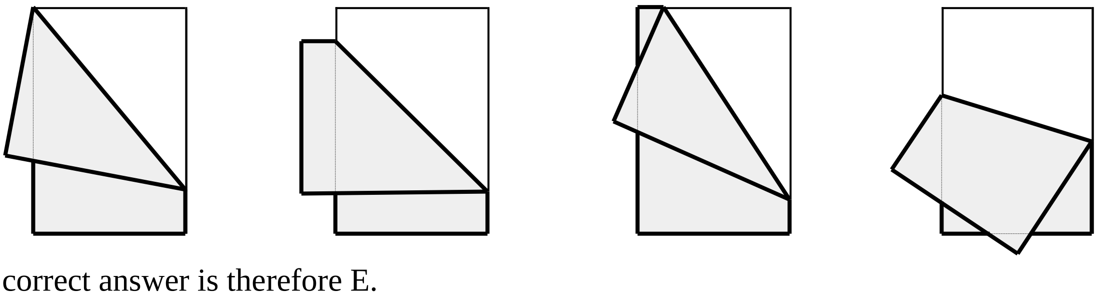
5) The diagram shows a triangular piece of paper that has been folded over to produce a shape with the outline of a pentagon. If a rectangular piece of paper is folded once, what is the smallest value of *n* (greater than 4) for which it is not possible to create an *n*-sided polygon in the same way?



A) 6 B) 7 C) 8 D) 9 E) 10

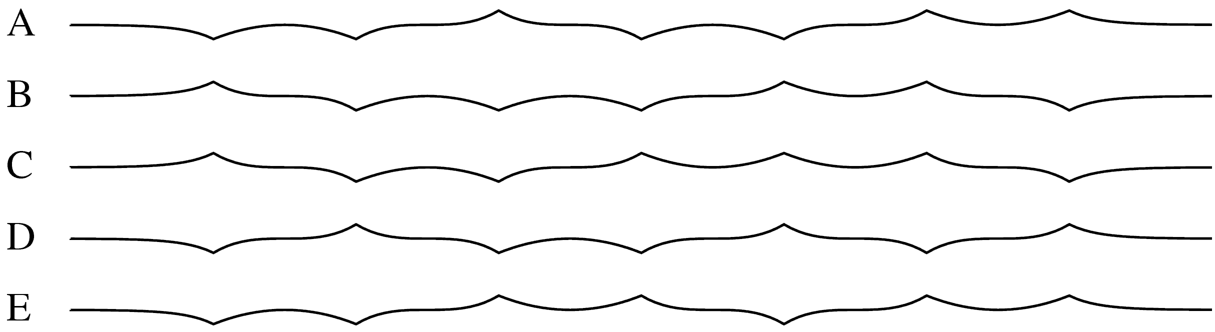
source: UK Senior Mathematical Challenge 2003, Nr. 10.

Solution: When the piece of paper is folded, the crease makes up one side of the resulting polygon. In addition, each of the four corners of the rectangle can contribute at most two sides to the resulting polygon. More than 9 sides are therefore certainly not possible. As we see in the figure, the other given values are possible.



The correct answer is therefore E.

6) A strip of paper is folded over in the middle three times. It is then unfolded and viewed from the side, so that all seven resulting folds can be viewed at once. Which of the following is not a possible result?

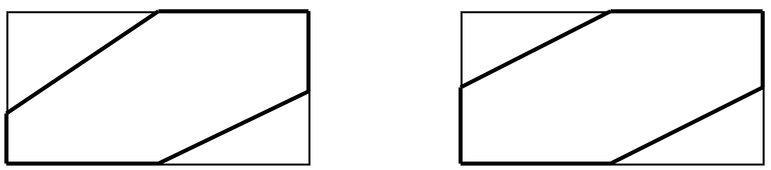


source: Kangaroo 2010, Junior

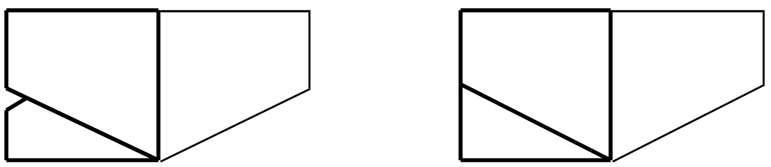
Solution: If folding a strip is to result in one of the shapes in the figure, the left half must be the same as the right, only oriented in the opposite direction. This is true of all five strips shown. However, the same must hold in the left (or right) half only; the left half of each half must be the same as the right, albeit upside-down. This is not the case for D, and this strip is therefore not possible. The correct answer is therefore D.

Comments

The first three of these problems are of the most elementary type. We are simply asked to imagine which shape can result by folding a rectangle once. An obvious idea for similar problems would be to consider a shape for the paper other than a rectangle. Also, any problem asking what is possible after two folds or more can become quite difficult to answer readily. Another interesting variation results from turning the situation around, and asking which shapes can be folded to make a certain shape. We can, for instance ask, which of the following shapes can be folded once in order to form a square.



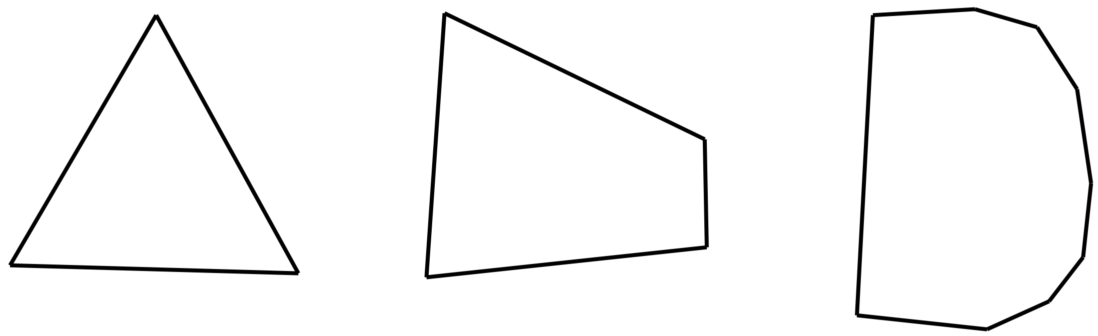
It may not be immediately obvious that it is possible for the hexagon on the right, but not for the one on the left, as we see below.



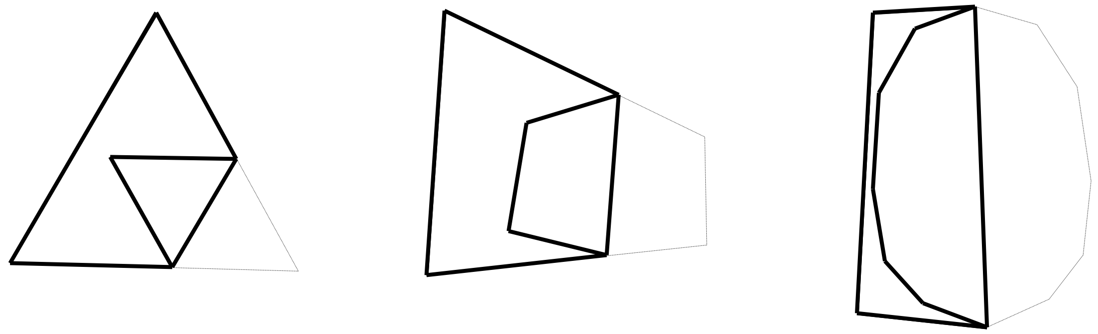
The difference is only in the placement of the end points of the sides of the hexagons on the left and right sides of the double squares. Similar variations are possible for the points on the top and bottom, of course.
A further step in generalization lets us ask which shapes can be folded twice to form a square, and many nice problems are waiting to be developed from this idea.

Problems 4 and 5 deal with *n*-gons that are folded to yield *m*-gons. This idea can be adapted in many ways. For instance, we can ask which values of m are possible if a specific value of n is given, or vice versa. In either case, we can either specify that the polygons are convex, or not. For specific values (*n*, *m* = 3, 4, …) this yields numerous questions, with some surprising answers. Let us consider one specific example.

Question: A convex *n*-gon is folded once. The result is a convex quadrilateral. Which values of *n* are possible?
Answer: All values $n \geq 3$ are possible. Consider the following figure:



We are given polygons with $n = 3$, $n = 4$, $n = 9$, the last of which is exemplary for any $n > 4$. Each can be folded to produce a quadrilateral, as we see below.

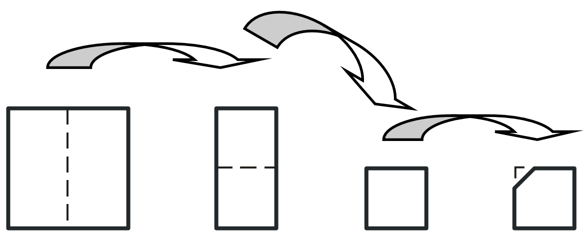


Finally, problem 6 is quite unusual. The interesting idea of viewing a flat folded (and unfolded) piece of paper from the side can certainly be used to produce a number of interesting problems. This particular problem has been shown to be very difficult for many people, children and adults alike. This may have something to do with the fact that many people simply find it difficult to reason in space, which is certainly necessary to solve this problem.

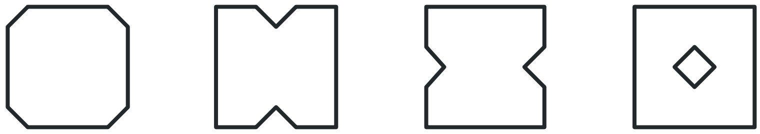
2. FOLD AND CUT

A small step leads us from considering the shapes that result from folding to shapes that result from folding, cutting, and unfolding again. Some problems of this type are the following.

7) A square piece of paper is folded twice as shown in the figure, so that a small square results. A corner of the resulting square is cut off and the square is unfolded again. Which of the following shapes cannot result is this way?



A) B) C) D)

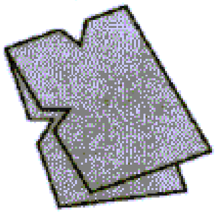


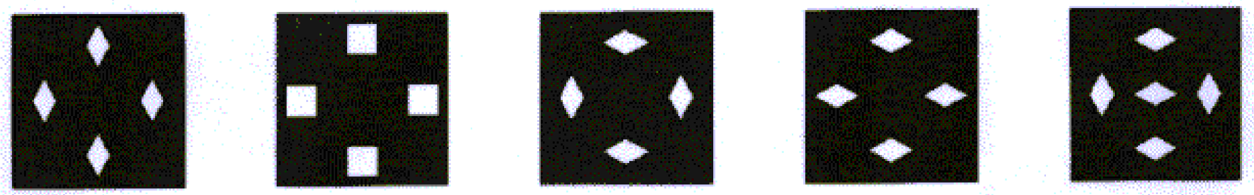
E) All shapes are possible.

source: Kangaroo 2007, Écoleier

Solution: All four shapes are possible. In fact, they correspond to the four corners of the small square being cut. The correct answer is therefore E.

8) This piece of paper was folded in half twice, and then had two equilateral triangles cut out of it. Which diagram shows how the paper will look when it is unfolded again?

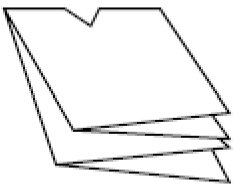




A) B) C) D) E)

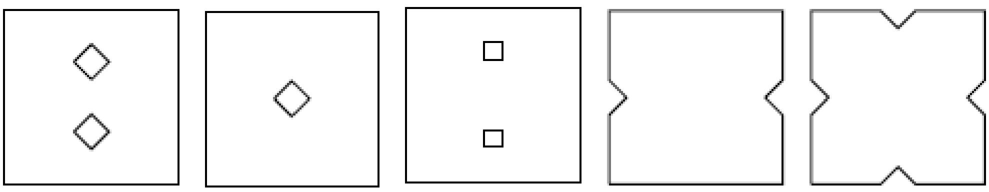
source: Kangaroo 2003, Cadet

Solution: The correct answer is obviously C.



9) Which is the result of unfolding the paper?

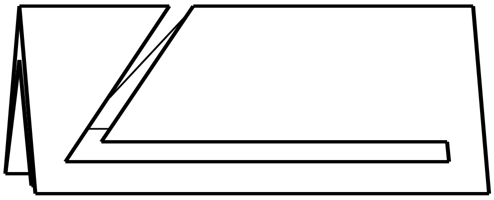
A) B) C) D) E)



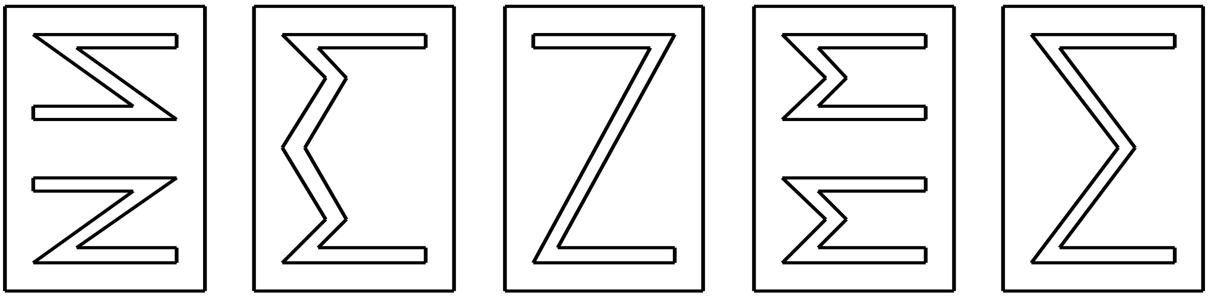
source: Kangaroo 2001, Benjamin

Solution: The answer is obviously A.

10) A rectangular sheet of paper is folded in half and then folded in half again. A piece is cut out of the paper, while folded, as shown. The sheet is then smoothed out to its original size again. Given that the pattern which appears is one of the following, which is it?



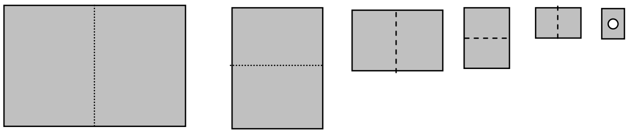
A) B) C) D) E)



source: AMC 2000

Solution: Since the paper was folded twice, the pattern formed on the paper must be symmetric with respect to a fold, and each half must again be symmetric with respect to a fold. Only D satisfies this requirement. The correct answer is therefore D.

11) Harold folds piece of paper five times as shown and pokes a hole through the folded paper. How many holes are there in the unfolded paper?



A) 6 B) 10 C) 16 D) 20 E) 32

source: Kangaroo 2004, Benjamin

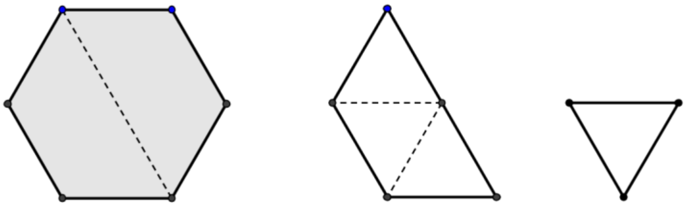
Solution: Each fold doubles the number of layers of paper. The final small rectangle is therefore made up of 32 layers. Since one hole is poked in each layer, the unfolded paper has 32 holes, and the correct answer is therefore E.

Comments

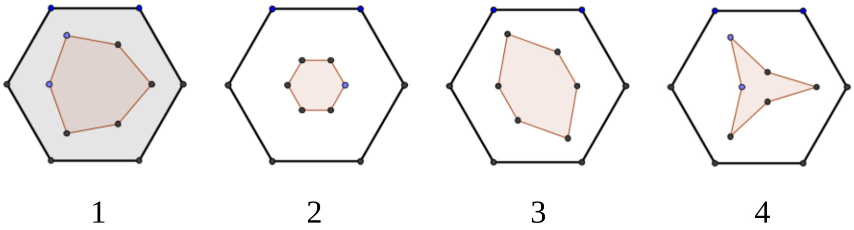
All of these problems involve a bit of spatial reasoning, and there is always a certain aspect of powers of 2 involved, since any fold always doubles the number of layers of paper being considered.
In problems 7 to 9, we are considering simple versions of "snowflakes", i.e. the shapes created by folding a piece of paper multiple times in such a way that a center of symmetry is created (together with the lines os symmetry resulting from the folds), making some cuts, and

then unfolding again, yielding a symmetric shape. It seems like an obvious variation to consider the hexagonal shape of a real snowflake. One such question could be the following:

Question: A piece of paper in the shape of a regular hexagon is folded over once and then into thirds as shown.



One straight cut is made, removing a section of the folded paper, and the remaining piece is unfolded again. Which of these shapes can be the result?



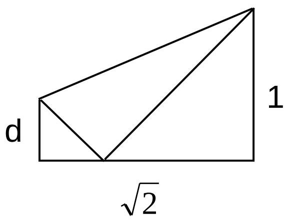
A) All are possible.
B) All except number 1 are possible.
C) All except number 2 are possible.
D) All except number 3 are possible.
E) All except number 4 are possible.

Answer: Because of the symmetry resulting from the folds, all but number 3 are possible. The correct answer is therefore D.

3. LENGTHS

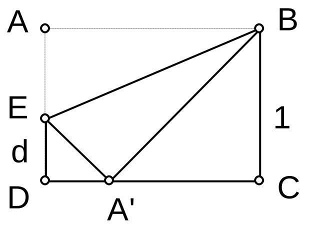
The most important aspect in most elementary questions concerning measurable attributes of flat folding, and certainly in all that involve only a single fold, is that of the resulting symmetry. „Folding over“ always implies leaving part of the paper stable on the folding surface, while reflecting the other part with respect to the folding edge.
Because of the resulting symmetry, the measures of angles and of lengths are retained, and this retention allows us to calculate a number of things that may, at first glance, appear not to be uniquely determined. This aspect is key to solving the questions posed in the following secrtions.
An important aspect in many problems concerning lengths specifically, is the use of the Pythagorean Theorem. This is, of course, to be expected if we are using a square (or rectangular) piece of paper, since the right angles in these are given. Even if we use some other shape as our starting point, however, right angles still turn up frequently, for instance if a line is folded onto itself. Right angles are therefore quite common, and calculation of lengths can therefore often be based on the calculation of the lengths of the sides of right triangles.
The following are some problems posed at competitions applying these ideas.

12) A rectangular sheet of paper with sides 1 and $\sqrt{2}$ has been folded once as shown, so that one corner just meets the opposite long edge.
What is the value of the length *d*?



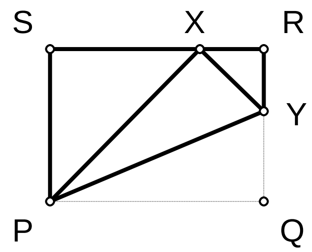
A) $\frac{1}{2}$ B) $\sqrt{2}-1$ C) $\frac{7}{16}$ D) $\sqrt{3}-\sqrt{2}$ E) $\frac{\sqrt{2}}{3}$

source: UK Intermediate Mathematical Challenge 1999, Nr. 25.



Solution: Since A folds onto A', we have A'B = AB = $\sqrt{2}$. by the Pythagorean Theorem, we therefore have A'C = 1. Since the triangle A'CB is right-angled and isosceles, so is A'DE, and we therefore have d = DE = DA' = $\sqrt{2}$ -1. The correct answer is therefore B.

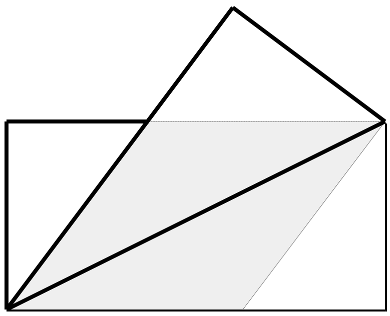
13) The rectangle PQRS represents a sheet of A4 paper, which means that PQ:PS = $\sqrt{2}$:1. The rectangle is folded as shown, so that Q comes to a point X on SR and the fold line PY passes through the corner P. Taking the length of PS to be 1 unit, find the lengths of the three sides of the triangle RXY.



source: UK Math Olympiad Hamilton Paper 2005, Nr. 5

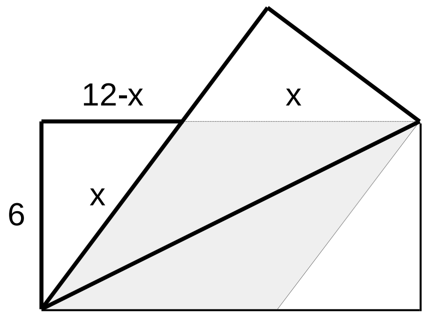
Solution: Since PX results from PQ, both segments are equally long. We therefore have PX = $\sqrt{2}$, and since PS = 1, the triangle SPX is therefore a right isosceles triangle. We therefore have SX = 1 and therefore also RX = $\sqrt{2}$ -1. Since the angle PXY is a right angle, it follows that the angle RXY is equal to SXP, and the triangle RXY is therefore also right isosceles. We therefore see that RY = RX = $\sqrt{2}$ -1 and XY = RX = $\sqrt{2}$ ($\sqrt{2}$ -1) = 2 - $\sqrt{2}$.

14) A rectangular piece of paper measuring 6 cm x 12 cm is folded along the diagonal. Those parts sticking out from the overlapping section are cut off and the remaining part unfolded. The result is a rhombus. How long are its sides?



A) $\frac{5}{2}\sqrt{5}$ cm B) 7.35 cm C) 7.5 cm D) 7.85 cm E) 8.1 cm

source: Kangaroo 2003, Junior



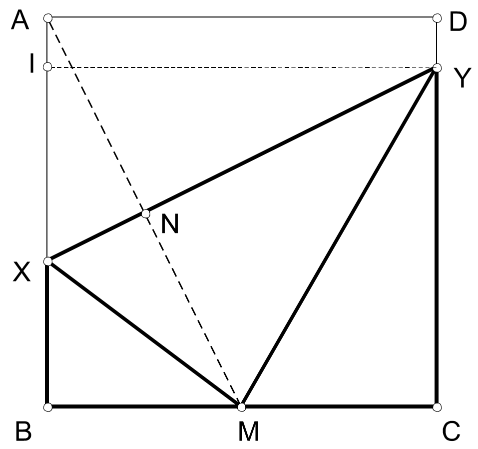
Solution: If we let x be equal to the length of the side of the rhombus, each of the right triangles we have cut off has sides of length 6, 12-x and x. We therefore have

$$(12-x)^2 + 6^2 = x^2,$$

and therefore x = 7.5 cm.
The correct answer is therefore C.

15) A square piece of paper *ABCD* is folded such that the corner *A* comes to lie on the mid-point *M* of the side *BC*. The resulting crease intersects *AB* in *X* and *CD* in *Y*. Show that $|AX| = 5\cdot |DY|$.

source: Mathematical Duel Bílovec – Chorzów – Graz – Přerov 2010



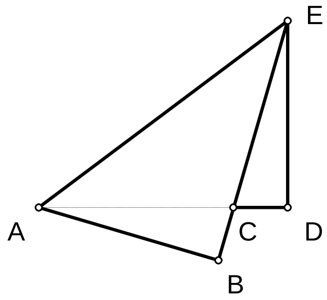
Solution: As shown in the figure, let *N* be the point of intersection of *AM* and the crease line *XY*. Furthermore, let the lengths of the sides of *ABCD* be equal to 1.

Since $|BM| = \frac{1}{2}$, we have $|AM| = \frac{\sqrt{5}}{2}$, and therefore $|AN| = \frac{1}{2}\cdot|AM| = \frac{\sqrt{5}}{4}$. The right-angled triangles *ABM* and *AXN* share the angle in *A*, and are therefore similar, and we have

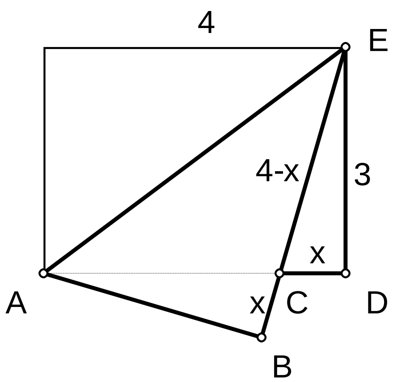
$$|AX|:|AN| = |AM|:|AB| \Leftrightarrow |AX|:\frac{\sqrt{5}}{4} = \frac{\sqrt{5}}{2}:1,$$

and therefore $|AX| = \frac{5}{8}$. If *I* is the foot of *Y* on *AB*, the triangles *YIX* and *ABM* are congruent, since their sides are pairwise orthogonal, and $|AB| = |YI| = 1$ holds. We therefore have $|IX| = |BM| = \frac{1}{2}$, and therefore $|DY| = |AX|-|IX| = \frac{5}{8}-\frac{1}{2} = \frac{1}{8}$, and we see that $|AX| = 5|DY|$ holds as claimed.

16) A 3 by 4 rectangle is folded along one of its diagonals to form the pentagon *ABCDE*, as shown. Calculate the perimeter of this pentagon.

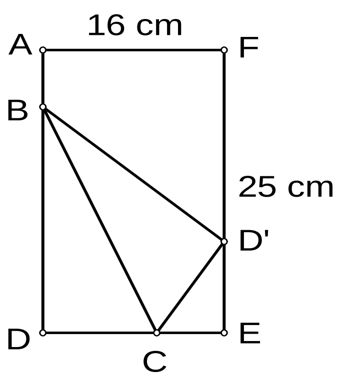


source: UKMT and Further Maths Network Senior Team Maths Challenge 2009, National final group round, Nr. 3



Solution: Due to symmetry, CD and CB are of the same length. Letting this length be denoted by x, we see that the sides of the triangle CDE are of the lengths x, 3 and 4-x, and be the Pythagorean theorem, we obtain $x^2 + 3^2 = (4-x)^2$, which yields $x = \frac{7}{8}$. Furthermore, ADE is also a right triangle, and since its sides are of length 3 and 4 respectively, the hypotenuse AE must be of length 5. The pentagon ABCDE therefore has two sides of length $x = \frac{7}{8}$, two of length 3 and one of length 5. Adding these values yields a perimeter of $12\frac{3}{4}$.

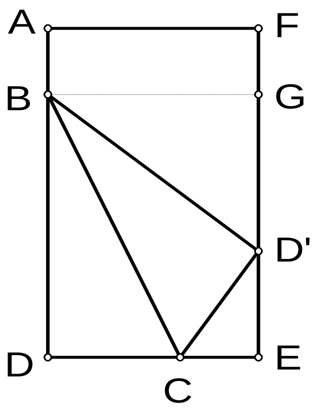
17) A 16 cm × 25 cm rectangular piece of paper is folded so that a corner touches the opposite side as shown. The distance AB = 5 cm. Find BC^2.



source: Australian Intermediate Mathematics Olympiad 2000, Nr. 6

Solution: We choose G on EF such that BG is parallel to AF. We then have BG = 16 cm, FG = 5 cm and GE = 25-5 = 20 cm. Since BD = BD' = 20, triangle BGD' yields

$$GD'^2 = BD'^2 - BG^2 = 400 – 256 = 144,$$



and we therefore have GD' = 12 cm, and therefore ED' = 20 – 12 = 8 cm.

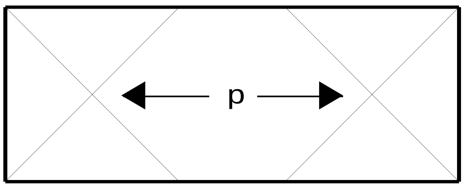
If we let CD = CD' = x, triangle CED' yields

$$x^2 = (16-x)^2 + 8^2,$$

and therefore x = 10 cm. Finally, triangle BCD therefore yields

$$BC^2 = BD^2 + CD^2 = 400 + 100 = 500.$$

18) A bus ticket is *m* cm long and *n* cm wide. During her bus trip Mary makes creases in the ticket as shown in the figure, where the creases bisect the angles in the corners. The length p, in centimetres, is



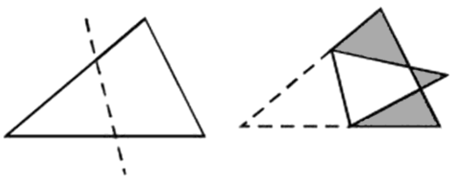
A) $m – 0.5n$ B) $m - 2n$ C) $m – n$ D) $m - \sqrt{2}\, n$ E) $\sqrt{2}\,(m – n)$

Source: AMC 1996

Solution: Since the intersection of the creases is at the center of a square of side *n*, the distance from the edge is $\frac{n}{2}$. The distance *p* is therefore equal to $m-2\cdot\frac{n}{2}=m-n$. The correct answer is therefore C.

4. AREA

19) A triangle is folded as shown in the figure. The area of the triangle is equal to 1.5 times the area of the resulting polygon. We know that the total area of the grey sections is equal to 1. Determine the area of the triangle.

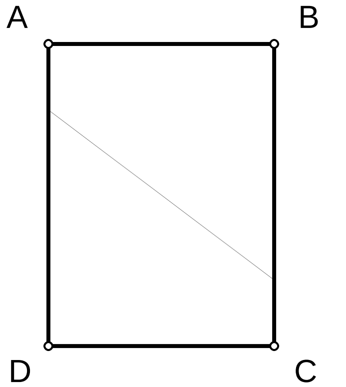


A) 2 B) 3 C) 4 D) 5 E) The area cannot be uniquely determined.

source: Kangaroo 2010, Junior

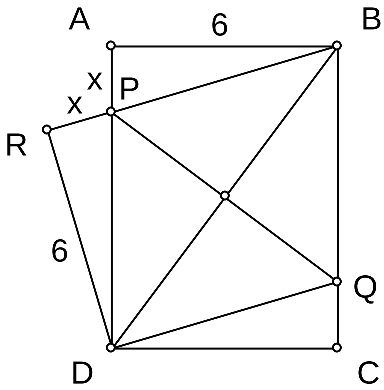
Solution: Let x be the area of the white quadrilateral in the right figure, i.e. of the part of the resulting polygon that is not grey. Since the small triangular piece on the left of the original triangle is composed of the tiny triangle on the right of the right hand figure together with this quadrilateral, the area of the original triangle is therefore equal to 1+2x. Since the area of the polygon is equal to 1+x, we have 1+2x = 1.5·(1+x), which yields x = 1. The area of the original triangle is therefore equal to 3, and the correct answer is therefore B.

20) A rectangular piece of paper ABCD in which AB = 6 cm and BC = 8 cm is folded over so that B folds exactly onto D and the folded paper is then laid flat on a table. What area of the table is then covered by the paper?



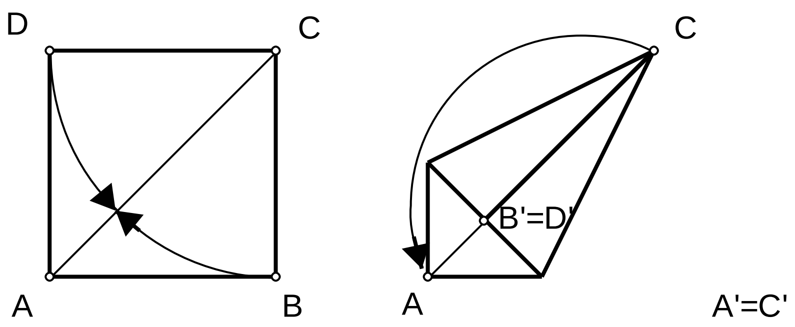
source: UK Math Olympiad Hamilton Paper 2003, Nr. B5

Solution: Let PQ be the crease resulting from the fold as shown in the diagram. Furthermore, let R be the point where A comes to rest after the fold. The area in question is therefore the area of the pentagon CQPRD. Also, let PA = PR = x.
The area we wish to calculate is the sum of the areas of the trapezoid PDCQ (which is half of the area of ABCD, and thus equal to (6·8):2 = 24 cm²) and the triangle PRD. Since the length of PD is equal to 8-x, applying the Pythagorean theorem in PRD gives us $x^2 + 6^2 = (8-x)^2$, which yields $x = \frac{7}{4}$. The area of PRD is therefore equal to $\frac{1}{2} \cdot 6 \cdot \frac{7}{4} = \frac{21}{4}$ cm², and the required area of the pentagon is therefore equal to $24 + \frac{21}{4} = 29\frac{1}{4}$ cm².



5. ANGLES

21) A pentagon was folded from a square of paper, as shown in the figure. At first the edges BC and DC were folded to the diagonal AC, so that the corners B and D lie on the diagonal and then the resulting shape was folded so that the vertex C coincided with the vertex A.

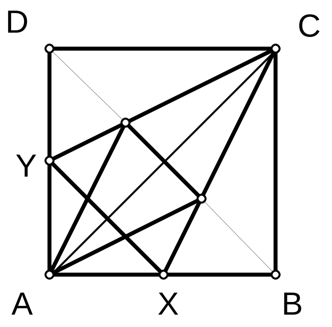


What is the size of the angle indicated by a question mark?

A) 104° B) 106.5° C) 108° D) 112.5° E) 114.5°

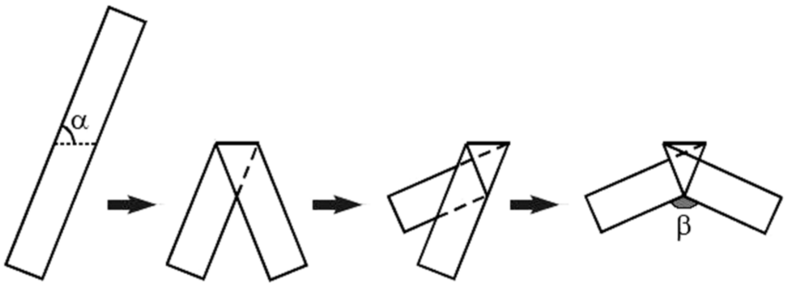
source: Kangaroo 2002, Cadet

Solution:
We first note that the crease from the last step lies on BD. Since BD is parallel to XY, the angle in question is equal to 180°-



∠CXY. Since BC is folded onto AC, the angle ∠XCA is half of ∠BCA,. and therefore equal to 22.5°. This means that ∠CXY is equal to 90°-∠XCA, and therefore to 67.5°. We therefore see that the angle in question is equal to 180°-∠CXY = 112.5°. The correct answer is therefore D.

22) A paper strip is folded three times as shown. Determine β if we are given that α = 70° holds.

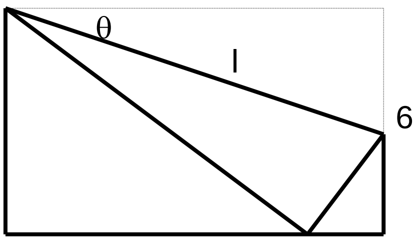


A) 140° B) 130° C) 120° D) 110° E) 100°

source: Kangaroo 2010, Étudiant

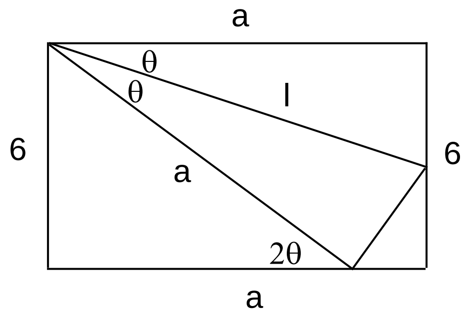
Solution: Since the angle 70° is folded down, the small triangle in the middle of the strip in the second figure is isosceles with angles of 70°, 70° and 40°. Folding to the left and right therefore yields angles of 140° to the left and right, which have an angle of 40° in common. the angle in question is therefore equal to 360°-(140°+140°-40°) = 120°. The correct answer is therefore C.

23) One side of a rectangular piece of paper is 6 cm and the adjacent sides are longer than 6 cm. One corner of the paper is folded so that it rests on the opposite longer side. If the length of the crease is *l* cm and it makes an angle θ with the position of the long side as shown, the *l* is

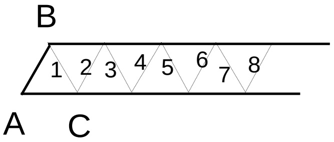


A) $6 \cdot \tan\theta$ B) $\dfrac{3}{\sin\theta\cos^2\theta}$ C) $\dfrac{6}{\sin^2\theta\cos\theta}$ D) $\dfrac{3}{\sin\theta\cos\theta}$ E) $\dfrac{3}{\sin^3\theta}$

source: AMC 1986

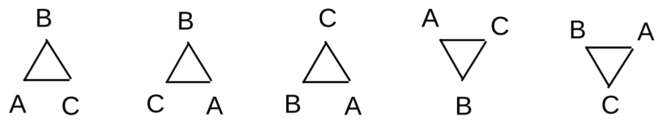


Solution: Since the right triangle (with angle θ) in the upper right-hand corner is folded down, we obtain another right triangle in the lower left-hand corner with angle 2θ. We therefore obtain $\sin 2\theta = \dfrac{6}{a}$ and $\cos\theta = \dfrac{a}{l}$. Multiplying these two equations yields $\dfrac{6}{l} = \dfrac{6}{a} \cdot \dfrac{a}{l} = \sin 2\theta \cdot \cos\theta = 2\sin\theta \cdot \cos\theta$, and therefore $l = \dfrac{3}{\sin\theta\cos^2\theta}$. The correct answer is therefore B.



24) As we see in the figure, a paper strip is divided into 2000 equilateral triangles along the dashed lines. The strip is successively folded at the lines in such a way that the strip always remains horizontal and the already folded part goes to the right. What is the position of triangle ABC after 1999 folds?

A) B) C) D) E)



source: Kangaroo 2000, Benjamin

Solution: The first fold brings the triangle into the position of answer E. Continuing the folding procedure, triangle ABC will be back in its original position after six folds. It will therefore also be back in the original position after 6·333 = 1998 folds. One more fold therefore returns it to the position after the first fold. The correct answer is therefore E.

25) Line *r* passes through the corner A of a sheet of paper and makes an angle α with the horizontal border, as shown in Figure 1. In order to divide α into three equal parts, we proceed as follows:

a) initially we mark two points B and C on the vertical border such that AB = BC; through B we draw a line *s* parallel to the border (Figure 2);

b) after that, we fold the sheet so as to make C coincide with a point C' on the line *r* and A with a point A' on line *s* (Figure 3); we call B' the point which coincides with B.

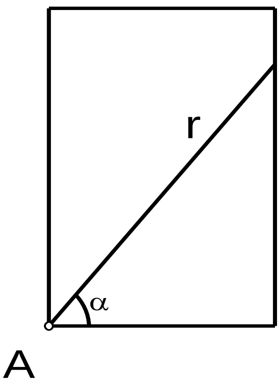


Figure 1

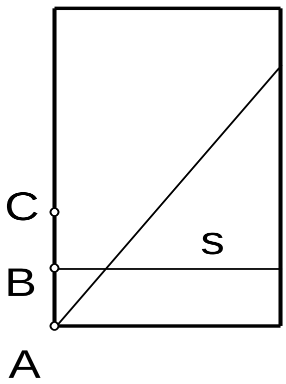


Figure 2

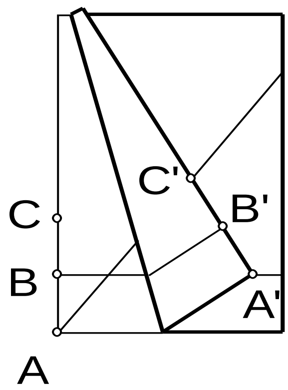
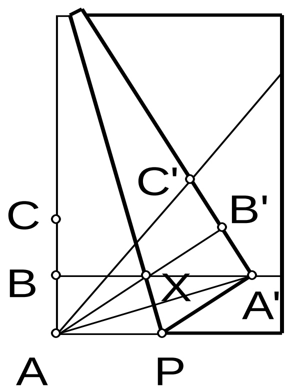


Figure 3

Show that lines AA' and AB' divide angle α into three equal parts.

source: 22nd Brazilian Mathematical Olympiad 2000, Nr. 1

Solution: Let P be the point in which the crease intersects the bottom edge of the paper and X the point in which the crease intersects with *s*. Furthermore, let β = ∠ PAA'. Since AP = AP', the triangle APA' is isosceles and we have ∠ PA'A = ∠ AA'P = β, and therefore ∠ PA'X = 2β, since AP and s are parallel. Since the triangles PAX and PA'X are congruent, we therefore also have ∠ PAX = 2β, and therefore ∠ XAA' = ∠ PAA' = β. Again, since AP and s are parallel, we therefore also have ∠ AXB = 2β, and thus ∠ A'XB' = 2β. Since AP and s are parallel, B' therefore lies on the

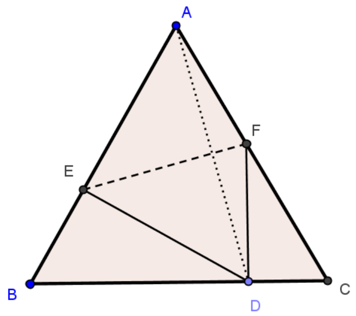


extension of AX. We see that AB’ is both altitude and median in the triangle AA’C’. It is an altitude since A’B is perpendicular to AC, and therefore AB’ must be perpendicular to A’C’, and it is a median because B is the mid-point of AC, and B’ must therefore be the mid-point of A’C’. We see that AA’C’ must be isosceles, and AB’ must also be the angle bisector in A, from which we deduce ∠ B’AC’ = ∠ B’AA’ = ∠ PAA’ = β, proving the claim.
Note that the angle trisection described here is well established in the origami math literature, and is due to H. Abe.

Comments

It is not difficult to find variations on the themes of these problems. Calculating the lengths of line segments, the angles between line segments and the areas of triangles or quadrilaterals is fairly standard stuff, and not usually interesting enough to be the kind of stuff we are looking for in a competition problem, however.
One possible idea for an interesting situation may be the following.

Question: We are given an equilateral triangle *ABC* with sides of unit length. The point *A* is folded to the point *D* on *BC* as shown, resulting in the crease *EF* with *E* on *AB* and *F* on *AC*. We assume that *FD* is perpendicular to *BC*.

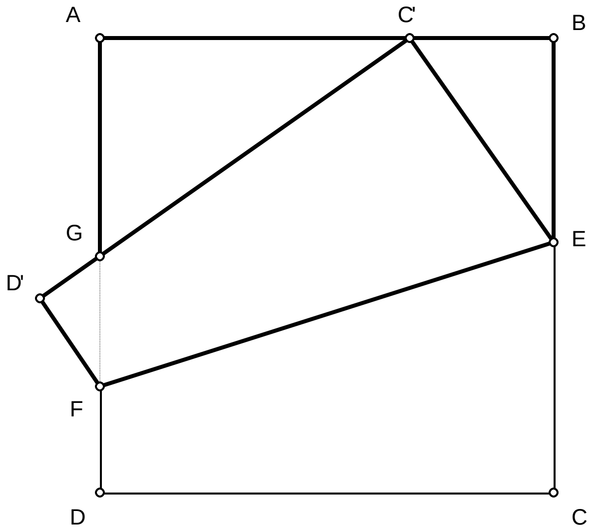


a) Determine the angle ∠ *AED*.
b) Determine the length of the line segment *CD*.
c) Determine the ratio of the areas of the triangles *AEF* and *ABC*.

Answer:
a) ∠ *AED* = 90°
b) |*CD*| = 2 - √3
c) [*AEF*]:[*ABC*] = (3√3 – 5):1

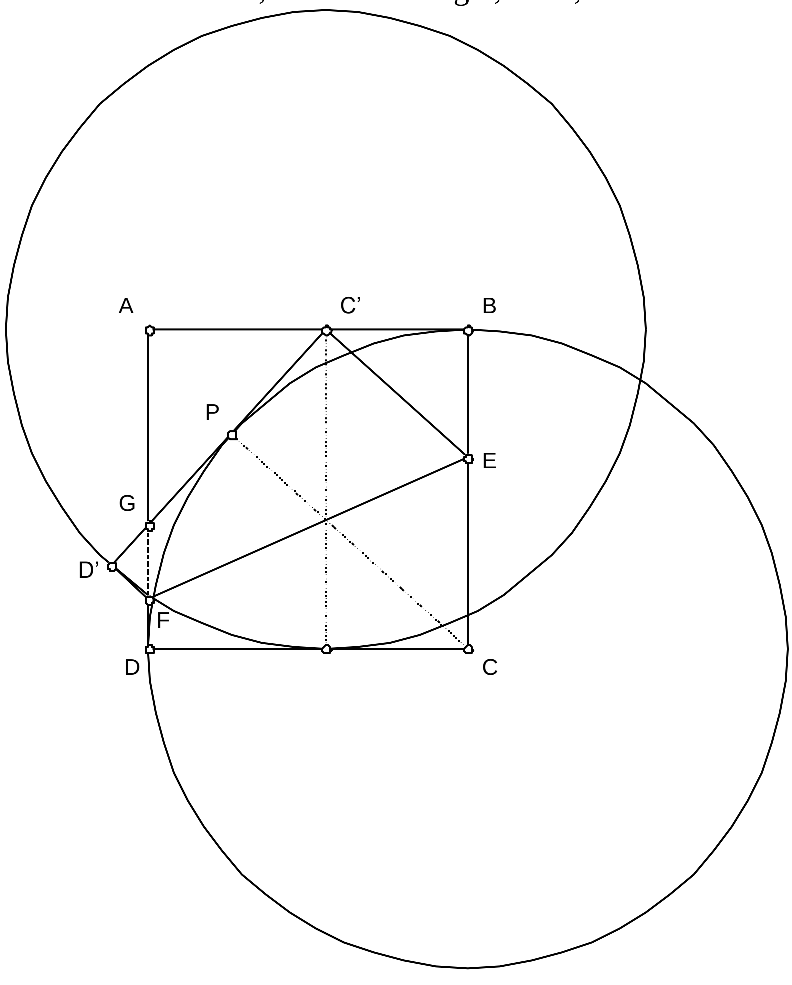
6. 6 FROM 1

In this section we consider problems resulting from the following situation. We are given a square piece of paper with corners A, B, C and D. Considering the corners A and B as fixed, we fold the paper such that the point C comes to lie on the edge AB. Corners C and D come to lie in positions C’ and D’ respectively. The resulting crease has end-points E and F on BC and DA respectively, and we use G to denote the common point of DA and C’D’.



26) Prove that C'D' is a tangent of the circle with center C. passing through B and D.

source: *More Mathematical Morsels*, Ross Honsberger, 1991, S.11



Solution: If we draw the circle with mid-point in C' and radius equal to the sides of the folding square, it is obvious that the circle is tangent to the bottom side of the square. Reflecting this circle with respect to EF yields the circle with mid-point in C through B and D, and reflecting CD with respect to EF yields C'D'. Since tangency is retained with the reflection, we have proven the claim to be true.

27) Prove that the perimeter of triangle GAC‘ is equal to half the perimeter of ABCD.

source: VIII Nordic Mathematical Contest 1994

Solution: We need only consider the fact that the tangent segments from a point to a circle are always of the same length. For this reason we have

AC‘ + C‘G + GA
= AC‘ + C‘P + GP + GA
= AC‘ + C‘B + GD + GA
= AB + DA,

and we are finished.

28) Prove the identity AG = C‘B + GD‘.

Solution: As before, we have

AC‘ + C‘G + GA
= AB + C‘D‘
= AC‘ + C‘B + C‘G + GD‘,

and therefore AG = C‘B + GD‘ holds as claimed.

29) Prove that the sum of the perimeters of triangles C‘BE and GD‘F is equal to the perimeter of triangle GAC‘.

source: 37th Slovenian Mathematical Olympiad 1993

Solution: We consider the similar triangles ΔGAC‘ ~ ΔC’BE ~ ΔGD’F. Since AG = C’B + GD’ holds, if follows that AC’ = BE + D’F and C’G = EC’ + FG also hold. We therefore have AG + AC’ + C’G = (C’B + BE + EC’) + (GD’ + D’F + FG), as claimed.

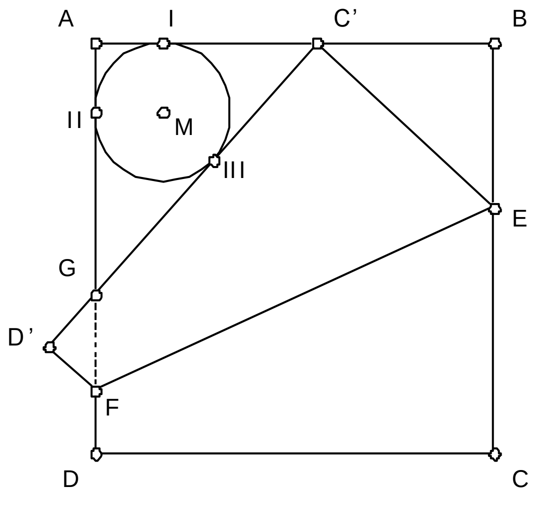
30) Prove that the perimeter of triangle GD‘F is equal to the length of line segment AC‘.

Solution: We need only consider once more the fact that the tangent segments from a point to a circle are always of the same length, and the fact that symmetric line segments are always of the same length. We therefore have AC‘ = D‘P = D‘G + GP = D‘G + GD = D‘G + GF + FD = D‘G + GD + FD‘, as claimed.

31) Prove that the inradius of GAC‘ is equal to the length of line segment GD‘.

source: traditional Sangaku problem from the Edo period from *Japanese Temple Geometry Problems*, Hidetoshi Fukagawa, 1989, S. 37

Solution: Naming C‘I = C‘III = x, GII = GIII = y, and AI = AII = r, we obtain
2·C‘D‘ = AC‘ + AG + GC‘ = (r + x) + (r + y) + (x + y) = 2·(x + y + r), and we therefore have
2·(x + y + GD‘) = 2·(x + y + r), and thus GD‘ = r, as claimed.



Comments

It is particularly interesting that the complete proof of the classic sangaku problem presupposes the proofs of the preliminary problems, two of which were actually posed at national or international competitions. Certainly, the extra problems 26, 28 and 30 could easily be used in a competition setting.
As a small side comment, it should be noted that paper folding problems are not common in sangaku. While some people may think of origami as quintessentially Japanese, the academic study of paper folding is only a fairly recent thing, the concept being almost unknown before the 20th century.

The following competitions yielded problems presented in this collection:

UK Mathematical Challenge
Australian Mathematics Competition (AMC)
Kangaroo Competiton
Australian Mathematics Olympiad
Brazilian Mathematical Olympiad
Slovenian Mathematical Olympiad
UK Math Olympiad
Mathematical Duel Bílovec – Chorzów – Graz – Přerov
Nordic Mathematical Contest UK Mathematical Challenge
American High School Mathematics Competition (AHSME)